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The Higgs Mechanism

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Contents

Chapter 1

Mini-Introduction to Quantum Field Theory

<u>Literature:</u> Textbooks:

[1] Bailin, David and Love, Alexander: *Introduction to gauge field theory*, Hilger

[2] Bjorken, James D. and Drell, Sidney D.: *Relativistische Quantenfeldtheorie*, BI-Wissenschaftsverlag.

[3] Böhm, M., Denner, A. and Joos, H.: *Gauge Theories of the Strong and Electroweak Interaction*, Teubner Verlag

[4] Cheng, Ta-Pei and Li, Ling-Fong: *Gauge theory of elementary partilce physics*, Oxford Science Publications

[5] Halzen, Francis and Martin, Alan D.: *Quarks and Leptons*, John Wiley & Sons, Inc.

[6] Itzykson, Claude and Zuber, Jean-Bernard: *Quantum Field Theory*, McGraw-Hill

[7] Kaku, Michio: *Quantum field theory*, Oxford University Press

[8] Kugo, Thaichiro: *Eichtheorie*, Springer

[9] Nachtmann, Otto: *Phänomene und Konzepte der Elementarteilchenphysik*, Vieweg

[10] Peskin, Michael E. and Schroeder, Daniel V.: *An introduction to quantum field theory*, Addison-Wesley

[11] Pokorski, Stefan: *Gauge field theories*, Cambridge University Press

[12] Ramond, Pierre: *Field theory*, Addison-Wesley

[13] Ryder, L.H.: *Quantum Field theory*, Cambridge University Press

[14] Sterman, George: *An Introduction to Quantum Field Theory*, Cambridge University Press

[15] Weinberg, Steven: *The quantum theory of fields*, Cambridge University Press

Web pages:

http://pdg.lbl.gov Particle Data Group

http://inspirehep.net Data base INSPIRE for publications

http://arxiv.org Preprint-Archive

http://www.cern.ch CERN

https://twiki.cern.ch/twiki/bin/view/AtlasPublic/HiggsPublicResults
ATLAS Public Higgs Results

https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResultsHIG
CMS Public Higgs Results

Throughout the lecture we will use natural units, *i.e.* we set the speed of light c equal to 1 and also Planck's constant is set to 1,

$$c = 1 \qquad \text{and} \qquad \hbar = 1 \; .$$

All physical units are then given in powers of the energy. The exponent is the (mass) dimension. We hence have

$$[\text{length}] = [\text{time}] = -1 \;, \quad [\text{mass}] = 1 \;, \quad [e] = 0 \; .$$

1.1 Langrangian Formalism for Fields

Starting from the Lagrangian function of a system of mass points

$$L = T - V \;, \tag{1.1}$$

with T and V denoting the kinetic and potential energy, respectively, we make the transition from discrete to continuous systems and get the Lagrangian density $\mathcal{L}$ as a function of q, dq/dt and $\vec{\nabla} q$. The canonical momentum is given by

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{q}} \; . \tag{1.2}$$

For three-dimensional systems we have

$$L = \int dx\, dy\, dz\, \mathcal{L} \; . \tag{1.3}$$

We apply Hamilton's principle to the Lagrangian density $\mathcal{L}(q, \dot{q}, \vec{\nabla} q)$. It says that the action S

$$S = \int_{t_1}^{t_2} dt \int d^3x\, \mathcal{L} \tag{1.4}$$

is minimized, while the boundary values $q(t_1)$, $q(t_2)$ are fixed. This leads to the Euler-Lagrange equation for fields

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \vec{\nabla} \frac{\partial \mathcal{L}}{\partial \vec{\nabla} q} = 0 \; . \tag{1.5}$$

The Hamiltonian density $\mathcal{H}$ is given by

$$\mathcal{H} = \pi\dot{q} - \mathcal{L} \;, \quad \text{with} \quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{q}} \;. \tag{1.6}$$

We define

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \quad \text{and} \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} \quad \text{and} \tag{1.7}$$

$$\int dx \equiv \int dt \int d^3x \;. \tag{1.8}$$

The $\int dx$ is Lorentz-invariant (the Lorentz contraction is compensated by the time dilatation). With this notation the field $\phi(t,\vec{x})$ is then denoted by $\phi(x)$ an the Lagrangian density by

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi) \;. \tag{1.9}$$

The Euler-Lagrange equations can then be written as

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \qquad \text{with} \quad \pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \;. \tag{1.10}$$

If the Lagrangian density $\mathcal{L}$ is Lorentz-invariant the field equations are covariant.

Let's have a look at the following example

1. Real scalar field without interaction. The Lagrangian density reads

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 \;. \tag{1.11}$$

We then have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \tag{1.12}$$

and hence the equation of motion

$$-m^2\phi - \partial_\mu \partial^\mu \phi = 0 \quad \Rightarrow \quad (\Box + m^2)\phi = 0 \quad \text{with } \partial_\mu \partial^\mu = \partial_0^2 - \vec{\nabla}^2 = \partial_0^2 - \Delta \;. \tag{1.13}$$

This it the Klein-Gordon equation known from relativistic quantum mechanics.

1.2 Noether-Theorem for Fields

Without proof

For each symmetry of the action with respect to a continuous transformation there exists a conservation law that can be derived from the Lagrangian density.

We look at $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$. Here φ is a field (a scalar field φ or $\varphi = A^\mu$ or a multiplet of fields $\varphi = (\varphi_1, ..., \varphi_n)$). We consider an infinitesimal transformation with respect to a Lie group

$$x^\mu \to {x'}^\mu = x^\mu + \delta x^\mu \tag{1.14}$$

with

$$\delta x^\mu = A_k^\mu \delta\omega^k \; . \tag{1.15}$$

Here $\delta\omega^k$ are the parameters of the transformation (*e.g.* the Euler rotation angles). For a rotation by $\delta\vec{\omega}$ $(\exp(i\delta\omega^k J_k))$ we have for example

$$\vec{x}' = \vec{x} + i\delta\omega^k J_k \vec{x} \; , \tag{1.16}$$

so that

$$A_k^0 = 0 \; , \quad A_k^j = i(J_k \vec{x})^j \; , j = 1,2,3 \; . \tag{1.17}$$

We furthermore have the transformation of the field

$$\varphi(x) \to \varphi'(x') = \varphi(x) + \delta\varphi(x) = \varphi(x) + \Phi_k(x)\delta\omega^k \; . \tag{1.18}$$

(For example we have for a scalar $\varphi'(x') = \varphi(x)$ and hence $\delta\varphi(x) = 0$, and for a vector $\varphi'^\mu(x') = \Lambda^\mu{}_\nu \varphi^\nu(x)$ etc.)

We talk about global transformations when $\delta\omega^k$ is independent of x. If S is invariant, *i.e.* $\delta S/\delta\omega_k = 0$, then

$$j_k^\mu = -\mathcal{L} A_k^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} (A_k^\nu \partial_\nu\varphi - \Phi_k(x)) \tag{1.19}$$

is a conserved current (Noether-Theorem):

$$\partial_\mu j_k^\mu = 0 \; . \tag{1.20}$$

We call j_k^μ Noether current. The charge

$$Q_k(x) = \int d^3x \, j_k^0(x) \tag{1.21}$$

is constant, because

$$\dot{Q}_k(x) = \int d^3x \, \partial_0 j_k^0 \overset{(1.20)}{=} -\int d^3x \, \vec{\nabla}\vec{j}_k = -\oint d\vec{S}\, \vec{j}_k = 0 \tag{1.22}$$

for $\vec{j}_k = 0$ at infinity. Note that the Noether current is not unique. Thus we can ad a current, $j_k'^\mu$, whose divergence vanishes, hence

$$\partial_\mu j_k'^\mu = 0 \; . \tag{1.23}$$

Der Noether current Eq. (1.19) can be generalised to the case of several fields φ^a in $\mathcal{L}$. We then have

$$j_k^\mu = -\mathcal{L} A_k^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^a)} (A_k^\nu \partial_\nu\varphi^a - \Phi_k^a(x)) \; . \tag{1.24}$$

<u>Examples</u>

We investigate the Noether theorem for an inner symmetry. We apply it for a complex field. The Lagrangian density reads

$$\mathcal{L} = \partial_\mu\varphi\partial^\mu\varphi^* - m|\varphi|^2 \; . \tag{1.25}$$

The Lagrangian density is invariant under a $U(1)$ symmetry, *i.e.* under the transformation

$$\varphi \rightarrow \varphi \exp(i\delta\vartheta) = \varphi + i\delta\vartheta\,\varphi \tag{1.26}$$

$$\delta x^\mu = 0 \quad \Rightarrow \quad A_k^\mu = 0 \tag{1.27}$$

$$\left.\begin{array}{rcl} \delta\varphi &=& i\delta\vartheta\,\varphi \\ \delta\varphi^* &=& -i\delta\vartheta\,\varphi^* \end{array}\right\} \overset{(1.18)}{\Rightarrow} \left(\begin{array}{c} \delta\varphi \\ \delta\varphi^* \end{array}\right) = \Phi\,\delta\vartheta\,, \qquad \text{with} \tag{1.28}$$

$$\Phi = \left(\begin{array}{c} i\varphi \\ -i\varphi^* \end{array}\right) = \left(\begin{array}{c} \Phi^1 \\ \Phi^2 \end{array}\right)\,. \tag{1.29}$$

The Noether current (1.19) reads

$$\begin{aligned} j^\mu &= -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\,\Phi^1 - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\,\Phi^2 \\ &= -i(\partial^\mu\varphi^*)\varphi + i(\partial^\mu\varphi)\varphi^*\,. \end{aligned} \tag{1.30}$$

The corresponding charge is given by

$$Q = \int d^3x\, j^0 = i\int d^3x\,(\varphi^*\dot{\varphi} - \varphi\dot{\varphi}^*)\,. \tag{1.31}$$

We consider the $U(1)$ symmetry of the Dirac theory. The Dirac Lagrangian density is invariant under the transformations

$$\psi \rightarrow \exp(i\theta)\,\psi\,, \qquad \bar{\psi} \rightarrow \exp(-i\theta)\,\bar{\psi}\,. \tag{1.32}$$

The Lagrangian density is given by

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + e\,\bar{\psi}A\!\!\!/\psi\,. \tag{1.33}$$

It contains the coupling to the electromagnetic field A^μ. We have for Φ

$$\Phi = \left(\begin{array}{c} i\psi_a \\ -i\bar{\psi}_a \end{array}\right) \qquad \text{with the spinor index } a = 1,2,3,4\,. \tag{1.34}$$

We then have with the Noether theorem

$$j^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)}\,i\psi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_a)}\,i\bar{\psi}_a = -i\bar{\psi}\gamma^\mu i\psi = \bar{\psi}\gamma^\mu\psi\,. \tag{1.35}$$

This is the $U(1)$ current density of the Dirac field. If ψ is the electron field, then $ej^\mu = e\bar{\psi}\gamma^\mu\psi$ is the electromagnetic current density. And

$$Q = e\int d^3x\, j^0 = e\int d^3x\,\bar{\psi}\gamma^0\psi = e\int d^3x\,\psi^\dagger\psi \tag{1.36}$$

is the conserved total charge. Hence $e\psi^\dagger\psi$ is the charge density.

1.3 Quantisation of Fields

Electrodynamics treats classical fields which fulfill Maxwell's equations. On the other hand Maxwell's equations describe the propagation of photons in the quantised theory. How can we combine both viewpoints?

Furthermore we want to be able to describe the production and annihilation of particles.

The searched formalism has to take into account that both the Klein Gordon equation and the Dirac equation can have states with negative energy, and that there is the spin-statistic relation.

Quantum field theory is the framework to describe scattering processes. Its predictions are experimentally confirmed.

Construction of Lagrangian densities

For the construction of Lagrangian densities we apply the following rules:

1) Define the fields which should be described by the theory. (Fields)

2) The Lagrangian density has the form

$$\mathcal{L}(x) = \sum_i g_i \mathcal{O}_i(x) \; . \tag{1.37}$$

The $\mathcal{O}_i$ are products of fields at the same point (locality). They transform as Lorentz scalars. Thereby also the action and the dynamics is relativistically invariant. The g_i are constants. Their mass dimension is chosen such that

$$[g_i \mathcal{O}_i] = 4 \; . \tag{1.38}$$

If the theory shall also possess inner symmetries, also the $\mathcal{O}_i(x)$ must be invariant under these symmetries. (Relativistic invariance and symmetries)

3) The Lagrangian $\mathcal{L}$ must contain derivatives ∂_μ of the fields. Otherwise the canonically conjugated momentum associated with the field would disappear and the Euler Lagrange Equation would be $\partial\mathcal{L}/\partial\phi = 0$ and there would be not development in time. Note, that sometimes it can be helpful to introduce auxiliary fields, on which no derivative acts and which hence have no dynamics. (Dynamics)

4) The mass dimension of the field products $\mathcal{O}_i$ must not be larger than four so that the theory is renormalizable. Note, that this requirement is not fundamental. It can be abandoned in 'effective quantum field theories'. (Renormalizability)

5) Furthermore, the Lagrangian must contain *all* terms, that are compatible with the requirements 2) and 4). (Completeness)

(i) Quantisation of the scalar field

In the following we only give results without derivation.

The goal is to quantise the scalar field which still fulfills the Klein Gordon equation

$$(\Box + m^2)\phi = 0 \; . \tag{1.39}$$

The field ϕ is then interpreted as operator. This leads in the end to the particle interpretation.

The classical field φ fulfills the Klein Gordon equation

$$(\Box + m^2)\varphi = 0 \tag{1.40}$$

Special solutions of this equation are given by plain waves of the form $\exp(ikx)$ or $\exp(-ikx)$, with $k^2 = m^2$. Hence

$$k_0 = \pm\sqrt{m^2 + \vec{k}^2} \equiv \pm\omega(\vec{k}) = \omega_k \ . \tag{1.41}$$

We then have

$$\Box \exp(ikx) = -k^2 \exp(ikx) \ . \tag{1.42}$$

The general solution is given by

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \Big(\alpha(\vec{k}) \exp(-ikx) + \alpha^*(\vec{k}) \exp(ikx) \Big) \ . \tag{1.43}$$

The factor

$$\frac{1}{(2\pi)^3 2\omega_k} \tag{1.44}$$

is convention. The measure

$$d^4k\, \delta(k^2 - m^2)\theta(k_0) = \frac{d^3k}{2\omega_k} \tag{1.45}$$

is Lorentz-invariant.

<u>Transition to the quantised field:</u> The Fourier coefficients are operators. Hence

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \Big(a(\vec{k}) \exp(-ikx) + a^\dagger(\vec{k}) \exp(ikx) \Big) \ . \tag{1.46}$$

The $a^\dagger$ can be interpreted as creation operator and a as annihilation operator. Applying $a^\dagger$ to the vacuum we get

$$|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle \ . \tag{1.47}$$

This is a 1-particle state with momentum $\vec{k}$. A particle is created in momentum space with the energy $k_0 = \sqrt{\vec{k}^2 + m^2}$. The whole Hilbert space ($\equiv$ Fock space) can then be constructed as:

$$|\vec{k}_1, \vec{k}_2\rangle = a^\dagger(\vec{k}_1)\, a^\dagger(\vec{k}_2)|0\rangle \tag{1.48}$$

$$|\vec{k}_1, ..., \vec{k}_n\rangle = a^\dagger(\vec{k}_1)...a^\dagger(\vec{k}_n)|0\rangle \ . \tag{1.49}$$

The latter is a state in Fock space, which consists of n particles with momenta $\vec{k}_i$.

The operators a and $a^\dagger$ fulfill the commutation relation

$$[a(\vec{k}_1), a^\dagger(\vec{k}_2)] = (2\pi)^3 2\omega_1 \delta(\vec{k}_1 - \vec{k}_2) \ . \tag{1.50}$$

And

$$[a^\dagger, a^\dagger] = 0 \quad (1.51)$$
$$[a, a] = 0 \,. \quad (1.52)$$

(ii) Charged scalar field

The field $\phi = \phi^\dagger$ describes self-conjugated particles, *i.e.* particles that are their own antiparticle. Examples are the neutral pion π^0 or the Higgs boson. But there are also spin-0 particles that are not their own antiparticle, like the charged pions π^+, π^- or the Kaon $K^0, \bar{K}^0$. These particles cannot be described by a Hermitean field. They are described by the field ($d\tilde{k} = d^3k/(2\pi)^3$)

$$\phi(x) = \int d\tilde{k}\,[a(\vec{k})e^{-ikx} + \underbrace{b^\dagger(\vec{k})}_{\neq a^\dagger,\text{ as }\phi\neq\phi^\dagger} e^{ikx}] \,. \quad (1.53)$$

The antiparticle is described by $\phi^\dagger$,

$$\phi^\dagger(x) = \int d\tilde{k}\,[b(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx}] \,. \quad (1.54)$$

We have the commutator relations

$$[a(\vec{k}), a^\dagger(\vec{k}')] = [b(\vec{k}), b^\dagger(\vec{k}')] = (2\pi)^3 2\omega\delta(\vec{k} - \vec{k}') \quad (1.55)$$
$$\text{all other commutators} = 0 \,. \quad (1.56)$$

The interpretation of the operators $a, a^\dagger, b, b^\dagger$ is

$a^\dagger$ creates a particle of type a with spin 0 and Mass m (1.57)
$b^\dagger$ creates a particle of type b with spin 0 and Mass m (1.58)
a annihilates a particle of type a with Spin 0 and Mass m (1.59)
b annihilates a particle of type b with Spin 0 and Mass m . (1.60)

Hence the field

ϕ annihilates a quantum of type a, creates a quantum of type b (1.61)
$\phi^\dagger$ annihilates a quantum of type b, creates a quantum of type a , (1.62)

and

$|a(\vec{k})\rangle$ is a 1-particle state with mass m, spin 0 and charge $+$ (1.63)
$|b(\vec{k})\rangle$ is a 1-particle state with mass m, spin 0 and charge $-$. (1.64)

These are the particle and its antiparticle.

(iii) Quantisation of spinor fields (Dirac fields)

The free Lagrangian density without interaction is given by

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \,, \qquad \text{where } \bar{\psi} = \psi^\dagger\gamma^0 \,. \quad (1.65)$$

The solution of the Dirac equation in quantised form is given by

$$\psi(x) = \int d\tilde{k} \sum_{s=\pm\frac{1}{2}} \left[\exp(ikx) b_s^\dagger(\vec{k})\, v_s(\vec{k}) + \exp(-ikx)\, a_s(\vec{k})\, u_s(\vec{k})\right] \quad (1.66)$$

$$\bar{\psi}(x) = \int d\tilde{k} \sum_{s=\pm\frac{1}{2}} \left[\exp(-ikx) b_s(\vec{k})\, \bar{v}_s(\vec{k}) + \exp(ikx)\, a_s^\dagger(\vec{k})\, \bar{u}_s(\vec{k})\right] \; . \quad (1.67)$$

And we have

$$a|0\rangle = b|0\rangle = 0 \; . \quad (1.68)$$

Therefore ψ creates an antiparticle (*e.g.* positron e^+) and annihilates a particle (*e.g.* electron e^-).

The operators fulfill anticommutation relations:

$$\{a_r(\vec{k}), a_s^\dagger(\vec{k}')\} = \delta_{rs}(2\pi)^3\, 2\omega_k\, \delta(\vec{k} - \vec{k}') \quad (1.69)$$

$$\{b_r(\vec{k}), b_s^\dagger(\vec{k}')\} = \delta_{rs}(2\pi)^3\, 2\omega_k\, \delta(\vec{k} - \vec{k}') \quad (1.70)$$

$$\{a, b\} = \{a, a\} = \{b, b\} = ... = 0 \; . \quad (1.71)$$

The 1-particle state

$$a_s^\dagger|0\rangle = |\vec{k}, s\rangle \quad (1.72)$$

is interpreted as electron with momentum $\vec{k}$ and spin s. The vacuum state is the state with $\vec{k} = \vec{0}$ and $s = 0$. Two-particle states are constructed through

$$|\vec{k}_1, s_1; \vec{k}_2, s_2\rangle = a_{s_1}^\dagger(\vec{k}_1) a_{s_2}^\dagger(\vec{k}_2)|0\rangle = -a_{s_2}^\dagger(\vec{k}_2) a_{s_1}^\dagger(\vec{k}_1)|0\rangle = -|\vec{k}_2, s_2; \vec{k}_1, s_1\rangle \; . \quad (1.73)$$

The Pauli principle holds. Spin-1/2-particles obey Fermi's statistics.

Chapter 2

Gauge symmetries

The Dirac Lagrangian for a free fermion fiels Ψ of mass m reads

$$\mathcal{L}_0 = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \, . \tag{2.1}$$

It is symmetric under $U(1)$, *i.e.* invariant under the transformations

$$\Psi(x) \to \exp(-i\alpha)\Psi(x) = \Psi - i\alpha\Psi + \mathcal{O}(\alpha^2) \, . \tag{2.2}$$

And for the adjoint spinor $\bar{\Psi} = \Psi^\dagger \gamma^0$ we have

$$\bar{\Psi}(x) \to \exp(i\alpha)\bar{\Psi}(x) \, . \tag{2.3}$$

The Noether current of this symmetry (see Eq. (1.35)) reads

$$j^\mu = \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \Psi)} \frac{\delta \Psi}{\delta \alpha} + \frac{\delta \bar{\Psi}}{\delta \alpha} \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \bar{\Psi})} = i\bar{\Psi}\gamma^\mu(-i\Psi) = \bar{\Psi}\gamma^\mu \Psi \, , \tag{2.4}$$

with

$$\partial_\mu j^\mu = 0 \, . \tag{2.5}$$

2.1 Coupling to a Photon

Taking into account the coupling to a photon, the Lagrangian reads

$$\mathcal{L} = \bar{\Psi}\gamma^\mu(i\partial_\mu - qA_\mu)\Psi - m\bar{\Psi}\Psi = \mathcal{L}_0 - qj^\mu A_\mu \, , \tag{2.6}$$

where j^μ has been given in Eq. (2.4). Performing a gauge transformation of the external photon field A_μ,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \tag{2.7}$$

the Lagrangian becomes

$$\mathcal{L} \to \mathcal{L} = \mathcal{L}_0 - qj^\mu A_\mu - \underbrace{qj^\mu \partial_\mu \Lambda}_{q\bar{\Psi}\gamma^\mu \Psi \partial_\mu \Lambda} \, . \tag{2.8}$$

This means that $\mathcal{L}$ is not gauge invariant. The gauge transformation of the fields Ψ and $\bar{\Psi}$ has to be changed such that the Lagrangian becomes gauge invariant. This can be done by the introduction of an x-dependent parameter α, hence $\alpha = \alpha(x)$. With this

$$i\partial_\mu \Psi \to i \exp(-i\alpha)\partial_\mu \Psi + \exp(-i\alpha)\Psi(\partial_\mu \alpha) \; , \tag{2.9}$$

so that

$$\mathcal{L}_0 \to \mathcal{L}_0 + \bar{\Psi}\gamma^\mu \Psi \partial_\mu \alpha \; . \tag{2.10}$$

This term cancels the additional term in Eq. (2.8) if

$$\alpha(x) = q\Lambda(x) \; . \tag{2.11}$$

The complete gauge transformation then reads

$$\Psi \;\to\; \Psi'(x) = U(x)\Psi(x) \qquad \text{with} \qquad U(x) = \exp(-iq\Lambda(x)) \quad (U \text{ unitary}) \tag{2.12}$$

$$\bar{\Psi} \;\to\; \bar{\Psi}'(x) = \bar{\Psi}(x)U^\dagger(x) \tag{2.13}$$

$$A_\mu(x) \;\to\; A_\mu(x) + \partial_\mu \Lambda(x) = U(x)A_\mu(x)U^\dagger(x) - \frac{i}{q}U(x)\partial_\mu U^\dagger(x) \; . \tag{2.14}$$

The Lagrangian transforms as ($U^\dagger = U^{-1}$)

$$\begin{aligned} \mathcal{L} \to \mathcal{L}' &= \bar{\Psi}\gamma^\mu U^{-1} i\partial_\mu (U\Psi) - q\bar{\Psi}U^{-1}\gamma^\mu \left(U A_\mu U^{-1} - \frac{i}{q} U \partial_\mu U^{-1} \right) U\Psi - m\bar{\Psi}U^{-1}U\Psi \\ &= \bar{\Psi}\gamma^\mu i\partial_\mu \Psi + \bar{\Psi}\gamma^\mu (U^{-1} i(\partial_\mu U))\Psi - q\bar{\Psi}\gamma^\mu \Psi A_\mu + \bar{\Psi}\gamma^\mu (i(\partial_\mu U^{-1})U)\Psi - m\bar{\Psi}\Psi \\ &= \mathcal{L} + i\bar{\Psi}\gamma^\mu \partial_\mu (\underbrace{U^{-1}U}_{\mathbb{1}})\Psi = \mathcal{L} \; . \end{aligned} \tag{2.15}$$

Minimal substitution $p_\mu \to p_\mu - qA_\mu$ leads to

$$i\partial_\mu \to i\partial_\mu - qA_\mu \equiv iD_\mu \; . \tag{2.16}$$

Here $D_\mu(x)$ is the *covariant derivative*. The expression *covariant* means, that it transforms exactly as the field

$$\Psi(x) \to U(x)\Psi(x) \qquad \text{und} \qquad D_\mu \Psi(x) \to U(x)(D_\mu \Psi(x)) \; . \tag{2.17}$$

This means

$$(D_\mu \Psi)' = D'_\mu \Psi' = D'_\mu U\Psi \stackrel{!}{=} UD_\mu \Psi \; , \tag{2.18}$$

so that the covariant derivative transforms as

$$D'_\mu = UD_\mu U^{-1} = \exp(-iq\Lambda)(\partial_\mu + iqA_\mu)\exp(iq\Lambda) = \partial_\mu + iq\partial_\mu \Lambda + iqA_\mu \stackrel{(2.7)}{=} \partial_\mu + iqA'_\mu \; . \tag{2.19}$$

With this

$$\mathcal{L} = \bar{\Psi}\gamma^\mu iD_\mu \Psi - m\bar{\Psi}\Psi \tag{2.20}$$

is manifestly gauge invariant.

The kinetic energy of the photons is given by

$$\mathcal{L}_{kin} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \text{mit} \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \; . \tag{2.21}$$

The field strength tensor $F^{\mu\nu}$ can be expressed through the covariant derivative (verify!): We choose the following ansatz for the tensor of 2nd rank

$$[D_\mu, D_\nu] = [\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu] = iq[\partial_\mu, A_\nu] + iq[A_\mu, \partial_\nu] = iq(\partial_\mu A_\nu - \partial_\nu A_\mu) \ . \tag{2.22}$$

With this we have for the field strength tensor

$$F^{\mu\nu} = \frac{-i}{q}[D^\mu, D^\nu] \ . \tag{2.23}$$

Its transformation behaviour is given by

$$\frac{-i}{q}[UD^\mu U^{-1}, UD^\nu U^{-1}] = \frac{-i}{q}U[D^\mu, D^\nu]U^{-1} = UF^{\mu\nu}U^{-1} \ . \tag{2.24}$$

2.2 Non-Abelian Gauge Groups

We have the Lagrangian for N Dirac fields ψ_i of mass m

$$\mathcal{L} = \sum_{j=1\ldots N} \bar{\psi}_j i\gamma^\mu \partial_\mu \psi_j - m \sum_{j=1\ldots N} \bar{\psi}_j \psi_j \ . \tag{2.25}$$

It is symmetric under $U(N)$, where $U(N)$ is the group of the unitary $N \times N$ matrices. We make the following transformation

$$\psi_j \to \sum_{k=1\ldots N} U_{jk}\psi_k \equiv U_{jk}\psi_k \ , \tag{2.26}$$

where in the last equation we have applied the Einstein sum convention. (= We sum over equal indices.) We then have

$$\Psi \to U\Psi \qquad \text{with} \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \ , \quad \text{hence} \quad \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \to \begin{pmatrix} U_{1k}\psi_k \\ U_{2k}\psi_k \\ \vdots \\ U_{Nk}\psi_k \end{pmatrix} \tag{2.27}$$

and

$$\mathcal{L} = \bar{\Psi} i\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi \to \bar{\Psi}U^{-1} i\gamma^\mu \partial_\mu U\Psi - m\bar{\Psi}U^{-1}U\Psi = \mathcal{L} \ . \tag{2.28}$$

Examples:

- $\Psi = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$: $SU_L(2)$, weak interaction acting on left-handed fermions.
- $\Psi = (q_1, q_2, q_3)^T$, quarks, $SU(3)$. Each of the q_i ($i = 1, 2, 3$) is a four-component spinor. The Lagrangian is invariant under $SU(3)$ transformations.

2.3 The matrices of the $SU(N)$

The elements of the $SU(N)$ can in general be represented by

$$U = \exp\left(i\theta^a \frac{\lambda^a}{2}\right) \qquad \text{mit} \qquad \theta^a \in \mathbb{R}\ . \tag{2.29}$$

The $\lambda^a/2$ are the generators of the group $SU(N)$. For the $SU(2)$ the λ^a are given by the Pauli matrices σ^i $(i = 1, 2, 3)$ and θ^a is a 3-component vector. For an element of the group $SU(2)$ we hence have

$$U = \exp\left(i\vec{\omega}\frac{\vec{\sigma}}{2}\right)\ . \tag{2.30}$$

For a general U we have

$$U^\dagger = \exp\left(-i\theta^a \left(\frac{\lambda^a}{2}\right)^\dagger\right) \stackrel{!}{=} U^{-1} = \exp\left(-i\theta^a \frac{\lambda^a}{2}\right)\ . \tag{2.31}$$

The generators hence have to be Hermitean, *i.e.*

$$(\lambda^a)^\dagger = \lambda^a\ . \tag{2.32}$$

Additionaly for the $SU(N)$ we require

$$\det(U) = 1\ . \tag{2.33}$$

With

$$\det(\exp(A)) = \exp(\mathrm{tr}(A)) \tag{2.34}$$

we get

$$\det\left(\exp\left(i\theta^a \frac{\lambda^a}{2}\right)\right) = \exp\left(i\theta^a \mathrm{tr}\left(\frac{\lambda^a}{2}\right)\right) \stackrel{!}{=} 1\ . \tag{2.35}$$

This leads to

$$\mathrm{tr}(\lambda^a) = 0\ . \tag{2.36}$$

The generators of the $SU(N)$ have to be traceless. The group $SU(N)$ has $N^2 - 1$ generators λ^a with $\mathrm{tr}(\lambda^a) = 0$. For the $SU(3)$ these are the Gell-Mann matrices

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\ .
\end{aligned} \tag{2.37}$$

The matrices $\lambda^a/2$ are normalized as

$$\mathrm{tr}\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right) = \frac{1}{2}\delta^{ab} \ . \tag{2.38}$$

For the Pauli matrices ($i = 1, 2, 3$) we have

$$\mathrm{tr}(\sigma_i^2) = 2 \qquad \text{und} \qquad \mathrm{tr}(\sigma_1\sigma_2) = \mathrm{tr}(i\sigma_3) = 0 \ . \tag{2.39}$$

Multiplied with 1/2 they are the generators of $SU(2)$.

2.4 Mass Terms and Symmetry Breaking

We look at the Lagrangian

$$\mathcal{L}_f = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \ . \tag{2.40}$$

In the chiral respresentation the 4×4 γ matrices are given by

$$\gamma^\mu = \left(\begin{pmatrix} \mathbf{0} & \mathbb{1} \\ \mathbb{1} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & -\vec{\sigma} \\ \vec{\sigma} & \mathbf{0} \end{pmatrix} \right) = \begin{pmatrix} 0 & \sigma_-^\mu \\ \sigma_+^\mu & 0 \end{pmatrix} \tag{2.41}$$

$$\gamma^5 = \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ \mathbf{0} & -\mathbb{1} \end{pmatrix} , \tag{2.42}$$

where σ_i ($i = 1, 2, 3$) are the Pauli matrices. With

$$\Psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} \quad \text{und} \quad \bar{\Psi} = \Psi^\dagger\gamma^0 = (\chi^\dagger, \varphi^\dagger) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (\varphi^\dagger, \chi^\dagger) \tag{2.43}$$

we get

$$\bar{\Psi} i\gamma^\mu D_\mu \Psi = i(\varphi^\dagger, \chi^\dagger) \underbrace{\begin{pmatrix} 0 & \sigma_-^\mu \\ \sigma_+^\mu & 0 \end{pmatrix} \begin{pmatrix} D_\mu\chi \\ D_\mu\varphi \end{pmatrix}}_{\begin{pmatrix} \sigma_-^\mu D_\mu\varphi \\ \sigma_+^\mu D_\mu\chi \end{pmatrix}} = \varphi^\dagger i\sigma_-^\mu D_\mu\varphi + \chi^\dagger i\sigma_+^\mu D_\mu\chi \ . \tag{2.44}$$

The gauge interaction holds independently for

$$\Psi_L = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \frac{1}{2}(\mathbb{1} - \gamma_5)\Psi \quad \text{and} \quad \Psi_R = \begin{pmatrix} \chi \\ 0 \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \gamma_5)\Psi \ . \tag{2.45}$$

The Ψ_L and Ψ_R can transform differently under gauge transformations,

$$\Psi'_L = U_L\Psi_L \quad \text{and} \quad \Psi'_R = U_R\Psi_R \ . \tag{2.46}$$

But

$$m\bar{\Psi}\Psi = m(\varphi^\dagger, \chi^\dagger) \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = m(\varphi^\dagger\chi + \chi^\dagger\varphi) = m(\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L) \ . \tag{2.47}$$

The mass term mixes Ψ_L and Ψ_R. From this follows *symmetry breaking* if Ψ_L and Ψ_R transform differently.

What about the mass term for gauge bosons. We have the Lagrangian

$$\mathcal{L} = -\frac{1}{4} \underbrace{F^{a\mu\nu} F^a_{\mu\nu}}_{\text{gauge invariant}} + \frac{m^2}{2} \underbrace{A^{a\mu} A^a_\mu}_{\text{not gauge invariant}} . \tag{2.48}$$

For example for the $U(1)$ we get

$$(A_\mu A^\mu)' = (A_\mu + \partial_\mu \theta)(A^\mu + \partial^\mu \theta) = A_\mu A^\mu + 2A_\mu \partial^\mu \theta + (\partial_\mu \theta)(\partial^\mu \theta) . \tag{2.49}$$

The mass term A^μ breaks the gauge symmetry.

Chapter 3

Spontaneous Symmetry Breaking

The symmetry of the Lagrangian is *spontaneously broken*, if the Lagrangian is symmetric, but the physical vacuum *does not* obey this symmetry. As consequence of the *Goldstone Theorem*, there are one or more massless spin 0 particles if the Lagrangian of the theory is invariant under an exact continuous symmetry which is not the symmetry of the physical vacuum These massless particles are called Goldstone bosons. In case the spontaneously broken symmetry is a local gauge symmetry, the interplay (induced by the Higgs mechanism) between the would-be Goldstone bosons and the massless gauge bosons leads to the masses of the gauge bosons and removes the Goldstone bosons from the spectrum.

3.1 Example: Ferromagnetism

We here have a system of interacting spins,

$$H = -\sum_{i,j} J_{ij} \vec{S}_i \cdot \vec{S}_j \; . \tag{3.1}$$

The scalar product of the spin operators is a singlet under rotations, hence rotationally invariant. In the ground state of the ferromagnet (at sufficiently low temperature, below the Curie temperature) all spins point in the same direction. This is the state of lowest energy. The ground state is no longer rotationally invariant. Upon rotation of the system a new ground state of same energy is reached, which differs, however, from the previous one. The ground hence is degenerate. The distinction of a specific direction breaks the symmetry. We have spontaneous symmetry breaking (SSB).

3.2 Example: Complex Scalar Field

We consider the Lagrangian for a complex scalar field

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 \quad \text{with the potential} \quad V = \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2 \; . \tag{3.2}$$

(Note: Adding higher powers in ϕ leads to a non-renormalizable theory.) The Lagrangian is invariant under a $U(1)$ symmetry.

$$\phi \to \exp(i\alpha) \phi \; . \tag{3.3}$$

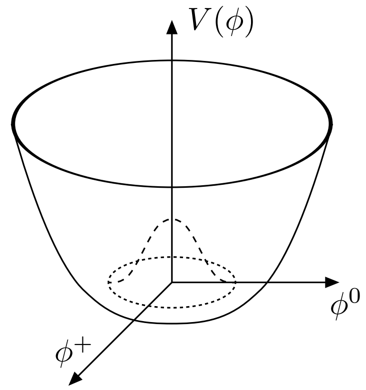

Figure 3.1: The Higgs potential.

We investigate the ground strate. It is given by the minimum of V,

$$0 = \frac{\partial V}{\partial \phi^*} = \mu^2 \phi + 2\lambda(\phi^*\phi)\phi \quad \Rightarrow \quad \phi = \begin{cases} 0 & \text{for} \quad \mu^2 > 0 \\ \phi^*\phi = -\frac{\mu^2}{2\lambda} & \text{for} \quad \mu^2 < 0 \end{cases} \tag{3.4}$$

The parameter λ has to be positive, so that the system does not become unstable. For $\mu^2 < 0$ the potential has the shape of a mexican hat, see Fig. 3.1. At $\phi = 0$ we have a local maximum, at

$$|\phi| = v = \sqrt{-\frac{\mu^2}{2\lambda}} \tag{3.5}$$

there is a global minimum. Particles correspond to harmonic oscillators for the expansion around the minimum of the potential. Fluctuations in the direction of the (infinitely many degenerate) minima have there is no inclination, so that they correspond to massless particles, the Goldstone bosons. Fluctuations perpendicular to this direction correspond to particles with mass $m > 0$. The expansion around the maximum at $\phi = 0$ would lead to particles with negative mass (tachyons), as here the curvature of the potential is negative.

Expansion about the minimum at $\phi = v$ leads to (we have for the complex field two fluctuations φ_1 and φ_2)

$$\phi = v + \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) = \left(v + \frac{1}{\sqrt{2}}\varphi_1\right) + i\frac{\varphi_2}{\sqrt{2}} \quad \Rightarrow \tag{3.6}$$

$$\phi^*\phi = v^2 + \sqrt{2}v\varphi_1 + \frac{1}{2}(\varphi_1^2 + \varphi_2^2) \; . \tag{3.7}$$

With this we get for the potential

$$V = \lambda(\phi^*\phi - v^2)^2 - \frac{\mu^4}{4\lambda^2} \qquad \text{with} \qquad v^2 = -\frac{\mu^2}{2\lambda} \quad \Rightarrow \tag{3.8}$$

$$V = \lambda\left(\sqrt{2}v\varphi_1 + \frac{1}{2}(\varphi_1^2 + \varphi_2^2)\right)^2 - \frac{\mu^4}{4\lambda^2} \; . \tag{3.9}$$

We can neglect the last term in V as it is only a constant shift of the zero point. We then get for the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1)^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 - 2\lambda v^2 \varphi_1^2 - \sqrt{2}v\lambda\varphi_1(\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2 \; . \tag{3.10}$$

The terms quadratic in the fields provide the masses, the terms cubic and quartic in the fields are the interaction terms. There is a massive and a massless particle,

$$m_{\varphi_1} = 2v\sqrt{\lambda} \quad \text{und} \quad m_{\varphi_2} = 0\ . \tag{3.11}$$

The massless particle corresponds to the Goldstone boson.

3.3 The Goldstone Theorem

We denote:

N	$=$	Dimension of the algebra of the symmetry group of the complete Lagrangian.
M	$=$	Dimension of the algebra of the group, under which the vacuum after the spontaneous symmetry breaking is invariant.

$\Rightarrow$ There are N-M Goldstone bosons without mass in the theory.

The Goldstone theorem says, that for each spontaneously broken degree of freedom there exists one massless Goldstone boson.

Chapter 4

Appendix

4.1 More Details on the Langrangian Formalism for Fields

The transition from discrete to continuous systems

To make the transition from the discrete to the continuous system we take the example of a chain of mass points. The mass points of mass m each are connected through springs with the spring constant k. Be a the average distance between two mass points and q_i the displacement of the i^{th} mass point from its position of rest. We then have the kinetic energy T

$$T = \sum_i \frac{1}{2} m \dot{q}_i^2 \; . \tag{4.1}$$

The potential energy V is given by

$$V = \sum_i \frac{1}{2} k (q_{i+1} - q_i)^2 \; . \tag{4.2}$$

With this the equation of motion for the i^{th} mass point reads

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i} = k(q_{i+1} - q_i) - k(q_i - q_{i-1}) \; . \tag{4.3}$$

On the other hand, the equation of motion can be derived from the Lagrangian function of the system. This is given by

$$L = T - V = \frac{1}{2} \sum_i a \left[\frac{m}{a} \dot{q}_i^2 - ka \left(\frac{q_{i+1} - q_i}{a} \right)^2 \right] \; . \tag{4.4}$$

The equation of motion for a single particle is obtained by applying the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \; , \tag{4.5}$$

leading to

$$\frac{m}{a} \ddot{q}_i - ka \, \frac{q_{i+1} - q_i}{a^2} + ka \, \frac{q_i - q_{i-1}}{a^2} = 0 \; . \tag{4.6}$$

Now we take the limit $a \to 0$. Here the following rules apply:

1. The quotient m/a becomes the mass density μ.
2. We have $\xi = (q_{i+1} - q_i)/a$ proportional zu to the force $k(q_{i+1} - q_i)$. The proportionality constant is given by the material constant y, the Young modul. We hence have

$$\frac{q_{i+1} - q_i}{a}\, y = k(q_{i+1} - q_i) \stackrel{a \to 0}{\to} y = k \cdot a \; . \tag{4.7}$$

3. We make the transition from the discrete index i to a continuous index x. Instead of the index i we now use the position of rest x. And instead of q_i we now have $q(x)$ as function of the position. We hence get

$$q_i \;\to\; q(x) \tag{4.8}$$

$$\frac{q_{i+1} - q_i}{a} \;\to\; \frac{q(x+a) - q(x)}{a} \stackrel{a \to 0}{\to} \frac{\partial q(x+a)}{\partial x} \approx \frac{\partial q(x)}{\partial x} = q'(x) \; . \tag{4.9}$$

Furthermore,

$$a \sum_i \to \int dx \; . \tag{4.10}$$

We thus obtain the following Lagrangian function of the continuous system

$$L = \int dx \left(\frac{1}{2}\, \mu\, \dot{q}(x)^2 - \frac{y}{2} \left(\frac{\partial q(x)}{\partial x} \right)^2 \right) \; . \tag{4.11}$$

The integrand is called Lagrangian density $\mathcal{L}$. The equation of motion is obtained from Eq. (4.6), yielding

$$\mu \ddot{q} - y \lim_{a \to 0} \left(\frac{q'(x+a) - q'(x)}{a} \right) = 0 \quad \Rightarrow \quad \mu \ddot{q} - y q'' = 0 \; . \tag{4.12}$$

Note, that x is <u>no</u> generalised coordinate, but an index. The canonical variable is given by $q(x) = q(t, \vec{x})$. We call $q = q(t, \vec{x})$ <u>field</u>. The equation of motion is a partial differential equation.

For three-dimensional systems we have

$$L = \int dx\, dy\, dz\; \mathcal{L} \; . \tag{4.13}$$

The Lagrangian density $\mathcal{L}$ is a function of $q, dq/dt$ and $\vec{\nabla} q$. The canonical momentum is given by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \; . \tag{4.14}$$

<u>The Euler-Lagrange Equation for Fields</u>

We apply Hamilton's principle to the Lagrangian density $\mathcal{L}(q, \dot{q}, \vec{\nabla} q)$. It says that the action S

$$S = \int_{t_1}^{t_2} dt \int d^3x\, \mathcal{L} \tag{4.15}$$

is minimized, while the boundary values $q(t_1)$, $q(t_2)$ are fixed. We have for the variation of S,

$$0 \stackrel{!}{=} \delta S = \int_{t_1}^{t_2} dt \int d^3x \, \frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} + \frac{\partial \mathcal{L}}{\partial (\vec{\nabla} q)} \delta(\vec{\nabla} q) \,, \text{ with } \delta \dot{q} = \frac{d}{dt} \delta q \,, \; \delta(\vec{\nabla} q) = \vec{\nabla} \delta q \tag{4.16}$$

A partial integration is performed while keeping the boundary terms fixed, so that their variation vanishes, *i.e.* $\delta q(t_1) = \delta q(t_2) = 0$. Furthermore, we demand that $q(t, \vec{x}) = 0$ for $|\vec{x}| \to \infty$. We then obtain

$$0 \stackrel{!}{=} \int_{t_1}^{t_2} dt \int d^3x \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \vec{\nabla} \frac{\partial \mathcal{L}}{\partial \vec{\nabla} q} \right) \delta q \tag{4.17}$$

This has to hold for all variations δq. We then get the Euler-Lagrange equation for fields

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \vec{\nabla} \frac{\partial \mathcal{L}}{\partial \vec{\nabla} q} = 0 \,. \tag{4.18}$$

The Hamiltonian density $\mathcal{H}$ is given by

$$\mathcal{H} = \pi \dot{q} - \mathcal{L} \,, \quad \text{with} \quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{q}} \,. \tag{4.19}$$

As an example we look at the following Lagrangian density

$$\mathcal{L} = \frac{\mu}{2} \dot{q}^2 - \frac{y}{2} q'^2 \,. \tag{4.20}$$

We have with

$$\frac{\partial \mathcal{L}}{\partial q} = 0 \,, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}} = \mu \dot{q} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial q'} = -y q' \tag{4.21}$$

the equation of motion

$$\mu \ddot{q} - y q'' = 0 \,. \tag{4.22}$$

<u>Relativistic Notation</u>
Further examples

1. Complex scalar field without interaction. The Lagrangian density reads

$$\mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*) = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \,. \tag{4.23}$$

The fields ϕ and ϕ^* can formally be varied independently of each other. This means that we have

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} = \partial^\mu \phi \tag{4.24}$$

$$\Rightarrow \partial_\mu \partial^\mu \phi + m^2 \phi = 0 \qquad \text{and analoguously} \quad \partial_\mu \partial^\mu \phi^* + m^2 \phi^* = 0 \,. \tag{4.25}$$

2. Spin-1/2 field (Dirac field) without interaction. The Lagrangian density reads

$$\mathcal{L}(\psi, \bar{\psi}) = \bar{\psi}(i\partial\!\!\!/ - m)\psi \; , \tag{4.26}$$

where

$$a\!\!\!/ \;:=\; a^\mu \gamma_\mu = a_\mu \gamma^\mu \tag{4.27}$$

$$\partial\!\!\!/ \;:=\; \gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma^\mu \partial_\mu \; . \tag{4.28}$$

We have

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\partial\!\!\!/ - m)\psi \qquad\qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0 \; . \tag{4.29}$$

With this then get the equation of motion

$$(i\partial\!\!\!/ - m)\psi = 0 \; . \tag{4.30}$$