From Elementary Particles to Nuclei and Their Interactions

Lecture II: Sub-Atomic Physics

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In This Presentation:

We keep discussing experiment-and-theory research strategies in Studying the Universe

After having discussed the century of discoveries related to elementary particles we will discuss the structure of us all:

The Systems of N-Identical Fermions

- Surprisingly, quarks will help us simplifying the complicated
- Quarks will force us to start new theories that 'ignore' them

How? Why? Just wait and see

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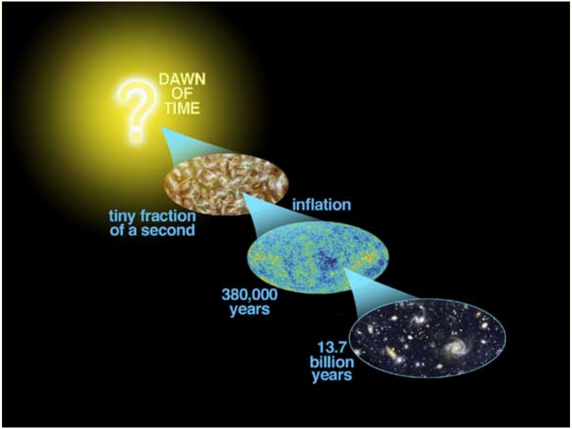
Part I

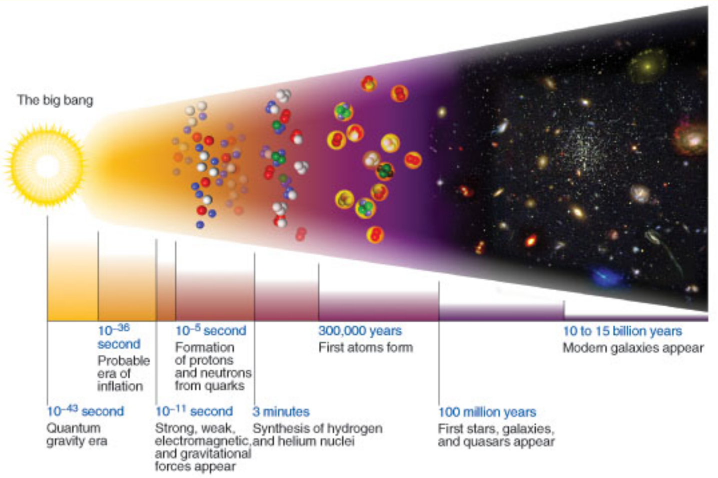
From Elementary Constituents of Matter to Sub-Atomic Systems: Atomic Nuclei

Cosmos: The Big Bang and Space Expansion

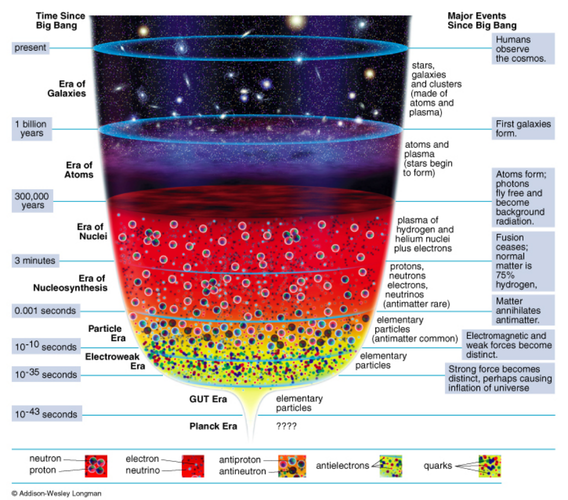


An artist's view of the Big Bang and space expansion that followed

Cosmos: Some Historical Turning-Points



Cosmos: Some Historical Turning-Points in Detail



Nucleosynthesis in Stars

Processes that Begin Nucleo-Synthesis

- **The first processes that begin an extremely rich nucleo-synthesis reaction sequence involve the only stable particles: the protons**

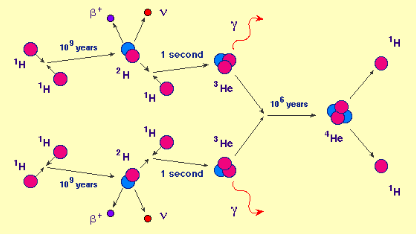
- **To overcome the Coulomb proton-proton repulsion, the very high temperatures of the stellar matter are needed**

The proton-proton chain:

$$^1_1\mathrm{H}_0 + ^1_1\mathrm{H}_0 \longrightarrow ^2_1\mathrm{H}_1 + \bar{\mathrm{e}} + \nu$$

$$^2_1\mathrm{H}_1 + ^1_1\mathrm{H}^1 \longrightarrow ^3_2\mathrm{He}_1 + \gamma$$

$$^3_2\mathrm{He}_1 + ^3_2\mathrm{He}_1 \longrightarrow ^4_2\mathrm{He}_2 + ^1_1\mathrm{H} + ^1_1\mathrm{H}$$



Processes with Participation of Heavier Nuclei

- **Once Helium nuclei are present the heavier ones can be produced**

$$ {}^4_2\mathrm{He}_2 + {}^4_2\mathrm{He}_2 \rightarrow {}^8_4\mathrm{Be}_4 + \gamma, \qquad {}^8_4\mathrm{Be}_2 + {}^4_2\mathrm{He}_2 \rightarrow {}^{12}_6\mathrm{C}_6 + \gamma \text{ etc.} $$

- **Example: CNO-Cycle**

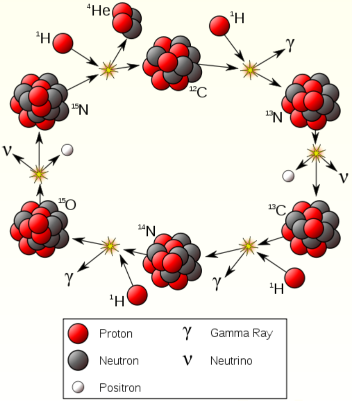
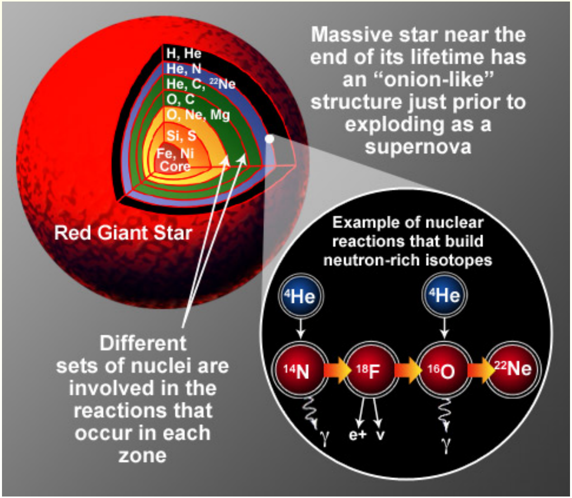
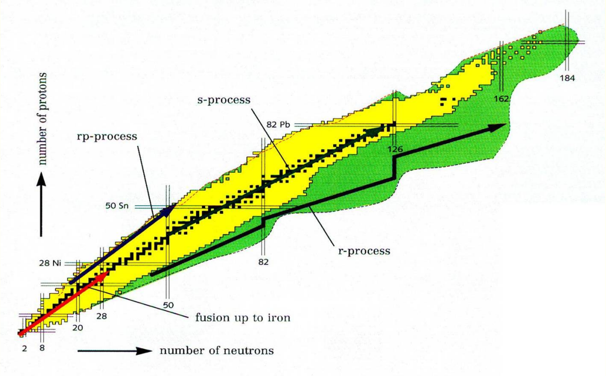


Illustration: Nucleo-Synthesis in Massive Stars



Nucleo-Synthesis: The Schemes of r- and s-Processes



The rapid (r) neutron capture process occurs in core-collapse of supernovae; produces neutron-rich atomic nuclei heavier than iron. The process beginns with iron 'seed' nuclei. The slow (s) neutron capture process, occurs in the so called AGB (asymptotic giant branch) stars. The central and inert core of AGB stars is composed of Carbon and Oxygen, surrounded by a shell where helium is undergoing fusion to form Carbon (Helium burning).

The s- and r-processes are believed to be mainly responsible for production of the elements heavier than iron.

Turning-Points in Cosmology

Cosmos: From Philosophy to Cosmology

Philosphy:

Pythagoras is said to have been the first philosopher to apply the term cosmos to the Universe

Theology:

The term Cosmos can be used to denote the created Universe, not including the creator

Cosmology:

Cosmology is the study of the cosmos in several of the above meanings, depending on context

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Cosmos: Some Historical Turning-Points

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- **In 1922:** Russian mathematician Alexander FRIEDMANN, derived the Friedmann equations based on Einstein's general relativity, and suggesting that the universe is very likely expanding
- **In 1924:** American astronomer Edwin HUBBLE shows that the nebulae of SLIPHER are in fact spiral galaxies
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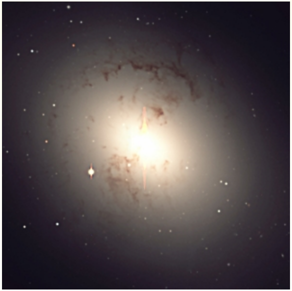
Examples of Galaxies Known Today

Astronomers classify galaxies in three main groups: elliptical, spiral and barred spiral. The elliptical ones range in shape from nearly spherical to highly flattened and contain from hundreds of millions to over one trillion stars.

This group encompasses both the biggest and the smallest of the known galaxies in the Universe. Some of the star-like objects in the field are globular clusters of stars that belong to the galaxy.

Credit: European Southern Observatory



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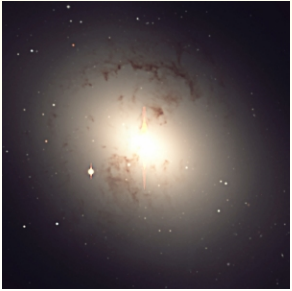

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Spiral galaxies consist of a flat, rotating disk containing stars, gas and dust, and a central concentration of stars known as the bulge. Spiral arms are regions of stars that extend from the center of spiral and barred spiral galaxies. A bulge is a huge, tightly packed group of stars.

Spiral arms contain many young, blue stars (due to the high mass density and the high rate of star formation). The bulk of the stars are located either close to a single plane or in a spheroidal galactic bulge around the galactic core.

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The bar is thought to channel gas inwards from the spiral arms and create new stars. The creation of the bar is generally thought to be the result of a density wave radiating from the center.

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• **In 1931:** Lemaître suggests that the expansion of the universe implies contraction backward in time implying Big-Bang hypothesis

• **In 1949:** An English astronomer Fred HOYLE, while criticising Lemaître's theory in a BBC radio broadcast uses the derisive phrase 'this big bang idea'... thus giving the name that is in use since then

• **In 1964:** Arno Penzias and Robert Wilson discover the so-called cosmic microwave background radiation, considered almost as an experimental proof of the Big Bang hypothesis (Nobel Prize in 1978)

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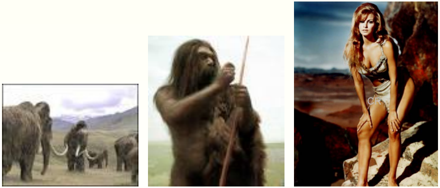
A Short Summary of Part I

- **Big Bang was an ultra-rapid space expansion rather than explosion**
- **Poorly known quantum-gravitational processes initiated Big Bang**
- **Within 10^{-35}-to-10^{-10} s strong & then electro-weak forces appear**
- **Within a next few minutes protons, neutron, electrons... are born**
- **Within 300 000 y atomic nuclei are formed (formation continues)**
- **Within 1 billion years the Galaxies are formed ... and slightly later:**



Artist's View

Part II

The Sub-Atomic Systems and Nucleon-Nucleon Interactions

Color Confinement → Quark Confinement

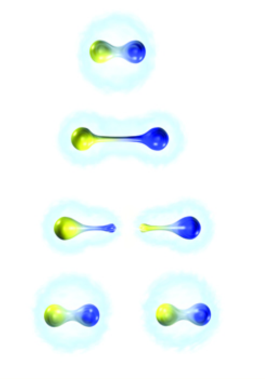
- **The forces acting between quarks increase with the distance, cf. Part I, and remain constant independently of the growing distance**

- **Quarks are surrounded by their gluon fields. Attempting to separate quarks by bringing-in more energy will just only increse the energy of the gluon field**

- **Gluons contribute to producing a new pair quark anti-quark (meson) rather than letting quark to be isolated**

- **Quarks remain in hadrons and never let themselves to be separated: we say**

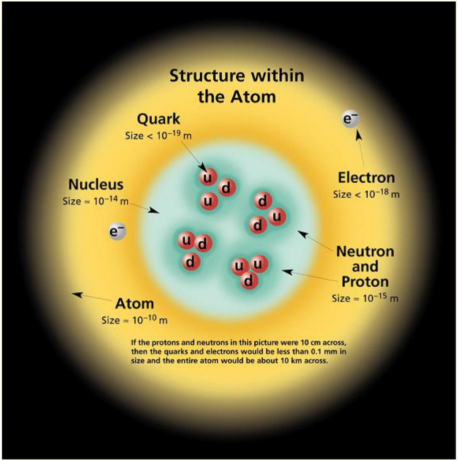
Quarks Remain Confined in Hadrons



Artist's View

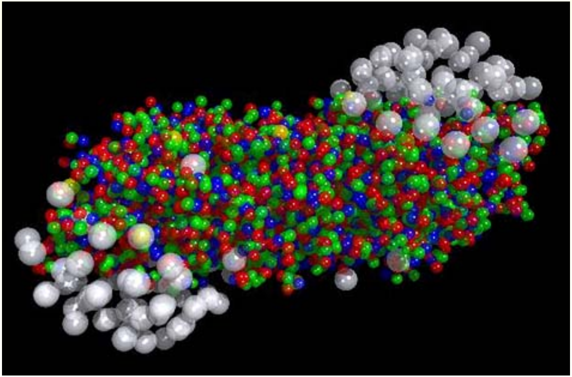
Confinement: Mathematical Formulation Simplifies

- **We will ignore the intrinsic quark structure of all nucleons**
- **Instead we will treat all of them as point particles**
- **Nucleon-Nucleon interactions will be mediated by pairs quark-antiquark but confined in mesons**
- **The formalism simplifies drastically and yet allows to perform <u>realistic</u> calculations for nuclei of interest**



High Energy Nuclear Collisions

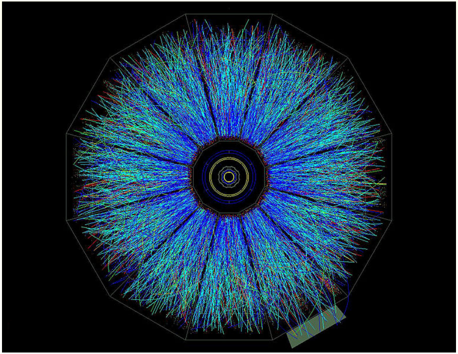
An ultra-high energy collision. The nucleons, grey bullets, are accompanied by growing number quarks (red, green and blue). The more energy - the more elementary constituents are produced $\rightarrow$ cf. next illustration



Artists's View of numerous virtual quarks surrounded by even more numerous gluons - not shown for artistic reasons

High Energy Nuclear Collisions

- **First Gold Beam-Beam Collision Events at RHIC at 100 - 100 GeV/c per beam recorded by the STAR detector. The tracks indicate the paths of thousands of subatomic particles as they pass through the STAR Time Projection Chamber, a large, 3-D digital camera.**



- **The presence of these particles indirectly proves the confinement mechanism**

Relativity and Symmetries

Relativity Principle and Space-Time Properties

- **According to Einstein's formulation* of the relativity principle:**

If a system of coordinates Σ is chosen so that, in relation to it, physical laws hold good in their simplest form, the same laws hold good in relation to any other system of coordinates Σ' moving in uniform translation relatively to Σ.

Albert Einstein: "The foundation of the general theory of relativity"

- **Uniformity of Space: All points in our 3D space are equivalent**
- **Isotropy of Space: All directions in our 3D space are equivalent**
- **Uniformity of Time: No time instant in our space is priviledged**

* In a non-relativistic approach as the one which follows we may use the historically earlier, Galilean formulation. For a form of a relativistic formulation cf. next Sections

Hamiltonian Form Allowed by Symmetries (1)

Hermiticity of the Hamiltonian. We must assume that the Hamiltonian is an observable and therefore Hermitean

$$\hat{H}(\hat{x}_1, \hat{x}_2) \equiv \hat{t}_1 + \hat{t}_2 + \hat{V}(\hat{x}_1, \hat{x}_2); \quad \hat{H}^\dagger = \hat{H} \;\rightarrow\; \hat{V}^\dagger = \hat{V}$$

Nucleons Are Indistinguishable. It follows that the Hamiltonian must be symmetric with respect to exchange of the two particles

$$\hat{H}(\hat{x}_1, \hat{x}_2) = \hat{H}(\hat{x}_2, \hat{x}_1) \;\rightarrow\; \hat{V}(\hat{x}_1, \hat{x}_2) = \hat{V}(\hat{x}_2, \hat{x}_1)$$

Translational Invariance. All reference frames are equivalent. Hamiltonians expressed in Σ and Σ' related to Σ by translation must be identical

$$\hat{V} = \hat{V}[(\hat{r}_1 - \hat{r}_2); (\hat{p}_1, \hat{p}_2); (\hat{s}_1, \hat{s}_2); (\hat{t}_1, \hat{t}_2)]$$

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$$\hat{H}(\hat{x}_1, \hat{x}_2) = \hat{H}(\hat{x}_2, \hat{x}_1) \;\rightarrow\; \hat{V}(\hat{x}_1, \hat{x}_2) = \hat{V}(\hat{x}_2, \hat{x}_1)$$

Translational Invariance. All reference frames are equivalent. Hamiltonians expressed in Σ and Σ' related to Σ by translation must be identical

$$\widehat{V} = \widehat{V}[(\hat{r}_1 - \hat{r}_2);\, (\hat{p}_1, \hat{p}_2);\, (\hat{s}_1, \hat{s}_2);\, (\hat{t}_1, \hat{t}_2)]$$

Hamiltonian Form Allowed by Symmetries (2)

Equivalence of Inertial Frames. Consider a reference frame Σ' moving with respect to Σ with *an arbitrary constant velocity* $\vec{v}$:

$$\boldsymbol{\Sigma}: \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\} \quad \longrightarrow \quad \boldsymbol{\Sigma}': \left.\begin{array}{l} \vec{\mathbf{v}}_1 \longrightarrow \vec{\mathbf{v}}_1' = \vec{\mathbf{v}}_1 + \vec{\mathbf{v}} \\ \vec{\mathbf{v}}_2 \longrightarrow \vec{\mathbf{v}}_2' = \vec{\mathbf{v}}_2 + \vec{\mathbf{v}} \end{array}\right\}$$

According to Galilean invariance, interactions expressed in either Σ or Σ' must be exactly the same and it follows that:

$$\hat{\mathbf{V}} = \hat{\mathbf{V}}[(\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2); (\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2); (\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2); (\hat{\mathbf{t}}_1, \hat{\mathbf{t}}_2)]$$

Notation. Introduce the relative positions $\hat{r}_{12}$ and relative momenta $\hat{p}_{12}$:

$$\hat{\mathbf{r}}_{12} \equiv \hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_1 \quad \text{and} \quad \hat{\mathbf{p}}_{12} \equiv \hat{\mathbf{p}}_2 - \hat{\mathbf{p}}_1$$

then:

$$\hat{\mathbf{V}} = \hat{\mathbf{V}}[\hat{\mathbf{r}}_{12}, \hat{\mathbf{p}}_{12}, (\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2); (\hat{\mathbf{t}}_1, \hat{\mathbf{t}}_2)]$$

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Hamiltonian Form Allowed by Symmetries (3)

Rotational Invariance. It is assumed that our space is isotropic and thus any two reference frames that differ by orientation must be equivalent. This implies that interaction potential must be constructed out of scalars

Examples: $\hat{\mathbf{r}}_{12} \cdot \hat{\mathbf{r}}_{12}, \quad \hat{\mathbf{p}}_{12} \cdot \hat{\mathbf{p}}_{12}, \quad \hat{\mathbf{r}}_{12} \cdot \hat{\mathbf{p}}_{12}, \quad \hat{\mathbf{r}}_{12} \cdot \hat{\mathbf{s}}_{12}, \quad \hat{\mathbf{p}}_{12} \cdot \hat{\mathbf{s}}_{12} \ldots$

Invariance Under Space Reflections. Since strong interactions conserve the parity, Hamiltonian must depend on scalars and not pseudo-scalars:

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Time-Reversal Invariance. We assume that the interaction Hamiltonian is time-reversal invariant. Recall:

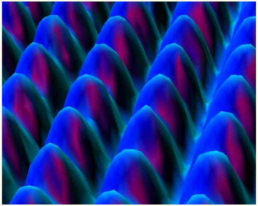
$$\hat{\mathbf{T}}\hat{\mathbf{r}}\hat{\mathbf{T}}^{-1} = +\hat{\mathbf{r}}, \quad \hat{\mathbf{T}}\hat{\mathbf{p}}\hat{\mathbf{T}}^{-1} = -\hat{\mathbf{p}}, \quad \hat{\mathbf{T}}\hat{\ell}\hat{\mathbf{T}}^{-1} = -\hat{\ell} \quad \text{and} \quad \hat{\mathbf{T}}\hat{\mathbf{s}}\hat{\mathbf{T}}^{-1} = -\hat{\mathbf{s}}$$

Hamiltonian must be contructed out of time-scalars, see a few examples:

$$\hat{\mathbf{p}}_{12} \cdot (\hat{\mathbf{s}}_1 + \hat{\mathbf{s}}_2), \quad (\hat{\ell}_1 + \hat{\ell}_2) \cdot (\hat{\mathbf{s}}_1 + \hat{\mathbf{s}}_2), \quad (\hat{\mathbf{r}}_{12} \wedge \hat{\mathbf{p}}_{12}) \cdot (\vec{\mathbf{s}}_1 + \vec{\mathbf{s}}_2) \ldots$$

Let Us Train Our Imagination of Space & Interaction

- **Hamiltonians of fundamental interactions are invariant under translations in any direction in any step**
- **Hamiltonians of fundamental interactions are invariant under any rotation through any angle**
- **They are simultaneously invariant under these and still some more transformations**



How Many Symmetries Do You Know?

General Hamiltonian Form Allowed by Symmetries

Concluding the Symmetry Considerations

• **Hamiltonians are: Hermitean and exchange-symmetric, Galilean symmetric, translation- and rotation-invariant, parity- and time-even**

• The symmetry considerations determine the forms of the simplest building-blocks, combinations of operators $\hat{r}$, $\hat{p}$, $\hat{s}$ and $\hat{t}$, see above

• The symmetries alone cannot determine the radial dependence of the interactions: for instance in a possible interaction operator

$$\hat{V}_{12} = v(r_{12})\,\hat{s}_1 \cdot \hat{s}_2$$

function $v(r_{12})$ remains undetermined by symmetry considerations

• Those functions must be determined by fitting to experimental results; fitting procedure remains phenomenological and not unique

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General Hamiltonian Form Allowed by Symmetries

Numerous experiments are compatible with the following forms of the Nucleon-Nucleon Interaction Hamiltonian

$\longrightarrow \longrightarrow \longrightarrow$

Fundamental Symmetries and Central Interaction

Denote $\hat{x} \stackrel{\text{df.}}{=} \{\hat{r}, \hat{p}, \hat{s}, \hat{t}\}$. Nuclear interactions have the form

$$\widehat{V}(\hat{x}_1, \hat{x}_2) \equiv \widehat{V}_C(\hat{x}_1, \hat{x}_2) + \widehat{V}_T(\hat{x}_1, \hat{x}_2) + \widehat{V}_{LS}(\hat{x}_1, \hat{x}_2) + \widehat{V}_{LL^2}(\hat{x}_1, \hat{x}_2)$$

where: C-central, T-tensor, LS-spin-orbit and LL^2-quadratic LS

Central Interaction ($r_{12} \equiv |\hat{r}_1 - \hat{r}_2|$)

$$\widehat{V}_C(\hat{x}_1, \hat{x}_2) = V_0(r_{12}) + V_s(r_{12})\,[\hat{s}^{(1)} \cdot \hat{s}^{(2)}] + V_t(r_{12})\,[\hat{t}^{(1)} \cdot \hat{t}^{(2)}] + V_{s-t}(r_{12})\,[\hat{s}^{(1)} \cdot \hat{s}^{(2)}]\,[\hat{t}^{(1)} \cdot \hat{t}^{(2)}]$$

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Fundamental Symmetries and Tensor Forces

Denote $\hat{x} \overset{df.}{=} \{\hat{r}, \hat{p}, \hat{s}, \hat{t}\}$. Nuclear interactions have the form

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where: C-central, T-tensor, LS-spin-orbit and LL^2-quadratic LS

Tensor Interaction [Non-Central]

$$\hat{S}^{(12)} \overset{df.}{=} \frac{3\,(\hat{s}_1 \cdot \hat{r}_{12})(\hat{s}_2 \cdot \hat{r}_{12}) - (\hat{s}_1 \cdot \hat{s}_2)\, r_{12}^2}{r_{12}^2} \quad and \quad r_{12} \overset{df.}{=} |\hat{r}_1 - \hat{r}_2|$$

$$\widehat{V}_T(\hat{x}_1, \hat{x}_2) = [V_{t_0}(r_{12}) + V_{t_1}(r_{12})\, \hat{t}_1 \cdot \hat{t}_2]\, \hat{S}^{(12)}$$

Invariant under rotations, translations, inversion and time-reversal

Fundamental Symmetries and Spin-Orbit Forces

Denote $\hat{x} \overset{df.}{=} \{\hat{r}, \hat{p}, \hat{s}, \hat{t}\}$. Nuclear interactions have the form

$$\boxed{\widehat{V}(\hat{x}_1, \hat{x}_2) \equiv \widehat{V}_C(\hat{x}_1, \hat{x}_2) + \widehat{V}_T(\hat{x}_1, \hat{x}_2) + \widehat{V}_{LS}(\hat{x}_1, \hat{x}_2) + \widehat{V}_{LL^2}(\hat{x}_1, \hat{x}_2)}$$

where: C-central, T-tensor, LS-spin-orbit and LL^2-quadratic LS

Spin-Orbit Interaction [Non-Local]

$$\hat{L} \overset{df.}{=} \frac{1}{2}(\hat{r}_1 - \hat{r}_2) \wedge (\hat{p}_1 - \hat{p}_2), \quad r_{12} \overset{df.}{=} |\hat{r}_1 - \hat{r}_2| \;\; and \;\; \hat{S} \overset{df.}{=} \hat{s}_1 + \hat{s}_2$$

$$\boxed{\widehat{V}_{LS}(\hat{x}_1, \hat{x}_2) = V_{LS}(r_{12})\, \hat{L} \cdot \hat{S}}$$

Invariant under rotations, translations, inversion and time-reversal

Fundamental Symmetries & Quadratic S-O Forces

Denote $\hat{x} \stackrel{df.}{=} \{\hat{r}, \hat{p}, \hat{s}, \hat{t}\}$. Nuclear interactions have the form

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where: C-central, T-tensor, LS-spin-orbit and LL^2-quadratic LS

Quadratic Spin-Orbit Interaction [Non-Local]

$$\hat{L} \stackrel{df.}{=} \frac{1}{2}(\hat{r}_1 - \hat{r}_2) \wedge (\hat{p}_1 - \hat{p}_2) \quad \text{and} \quad r_{12} \stackrel{df.}{=} |\hat{r}_1 - \hat{r}_2|$$

$$\widehat{V}_{LL}(\hat{x}_1, \hat{x}_2) = V_{LL}(r_{12})\{(\hat{s}_1 \cdot \hat{s}_2)\,\hat{L}^2 - \tfrac{1}{2}[(\hat{s}_1 \cdot \hat{L})(\hat{s}_2 \cdot \hat{L}) + (\hat{s}_2 \cdot \hat{L})(\hat{s}_1 \cdot \hat{L})]\}$$

Invariant under rotations, translations, inversion and time-reversal

... a Growing Evidence for 3-Nucleon Forces

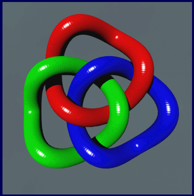
Nuclear three-body interactions have the general form

$$\boxed{\hat{V} = \hat{V}(\hat{x}_1, \hat{x}_2, \hat{x}_3)}$$

if they cannot be obtained as a sum of any two-body interactions

A Borromean Geometry

- The name "Borromean rings" comes from their use in the coat of arms of the aristocratic Borromeo family in Italy
- No two rings intersect: by cutting one of them the others fall apart



A Short Summary of Part II

- **Quark confinement allows to describe nucleons as point particles**
- **The quark sizes are four orders smaller than the sizes of nucleons**
- **The fundamental symmetries leave all the radial form-factors free**
- **The symmetries are: uniformity of space and time, isotropy of space, relativity principle, hermiticity of $\hat{H}$ and exchange symmetry**
- **We have central, tensor, spin-orbit and quadratic spin-orbit terms**
- **The Nuclear Interactions are in general non-central and non-local**
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- **Detailed discussion of the nuclear shell-model (2-body) tomorrow**

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Part III

Relativistic Mean-Field Theory

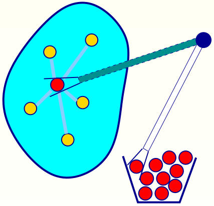
A Few Remarks about the Mean-Field Concept

- **A mean-field interaction can be seen as an algorithm probing the two-body interactions through a generalized weighted average $\widehat{V}$**

$$\widehat{V}(\hat{x}) = \frac{1}{N-1} \sum_{j=1}^{(N-1)} \int dx_j \psi^*(x_j)\, \widehat{V}(\hat{x}, \hat{x}_j)\, \psi(x_j)$$

- **Obseve that summation implies the averaging over (N-1)-particles**
- **Notice also that the mean-potential $\widehat{V} = \widehat{V}(\hat{x})$ is a one-body operator**
- **Relativistic theory illustrated in the following provides a similar concept but using quantum field theory**

An N–Body System



Schematic: Probing 2-body interactions with an 'external' test-particle

Quark confinement allows to use one-nucleon formalism approximation

Confinement & Low Energy Sub-Atomic Phenomena

- **In analogy to quantum electrodynamics whose Lagrangian-density***

$$\mathcal{L}_{\rm QED} = \mathcal{L}_{\rm Dirac} + \mathcal{L}_{\rm Maxwell} + \mathcal{L}_{\rm int}$$

or more explicitly

$$\mathcal{L}_{\rm QED} = \bar{\psi}(i\gamma^\mu p_\mu - m)\psi - \tfrac{1}{4}[F_{\mu\nu}]^2 + e(\bar{\psi}\gamma^\mu\psi)A_\mu$$

- **... we may introduce the so-called Yukawa interaction density:**

$$\mathcal{L}_{\rm Yukawa} = \mathcal{L}_{\rm Dirac} + \mathcal{L}_{\rm Klein-Gordon} - g(\bar{\psi}\psi)\phi$$

- **In subatomic physics this theory leads to coupled systems of the relativistic equations of the form:**

$$\left.\begin{array}{lcl} [\text{Dirac Equations for Nucleons}] & = & [\text{Nucleons Coupled with Mesons}] \\ [\text{Klein}-\text{Gordon Eqs for Mesons}] & = & [\text{Mesons Coupled with Nucleons}] \end{array}\right\}$$

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• **In such theories we obtain the Dirac-type wave-equations for the nuclens moving in the average fields of all other particles ...**

• **... and the Klein-Gordon-type wave equations for mesons moving in the average fields of all other particles**

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Just a Short Reminder abot Dirac Equation

- **The so-called covariant form of the free Dirac equation reads***

$$(\gamma^\mu \hat{p}_\mu - m\,c)\,\psi = 0; \quad \{\hat{p}_\mu\} \equiv \left\{ i\left(\frac{\hbar}{c}\right)\frac{\partial}{\partial t},\, i\hbar\hat{\nabla} \right\}$$

- **Schrödinger-like form of the free Dirac equation (just insert $\hat{p}_\mu$)**

$$i\hbar\,\frac{\partial\psi}{\partial t} = -i\hbar c\left(\hat{\alpha}\cdot\hat{\nabla}\right)\psi + \beta\left(mc^2\right)\psi; \quad \psi \sim \varphi\, e^{\pm i\,\frac{\mathcal{E}\,t}{\hbar}}$$

- **Equivalent stationary form of the free Dirac equation:**

$$\left[-i\hbar c\left(\hat{\alpha}\cdot\hat{\nabla}\right) + \beta\,(mc^2)\right]\varphi = \mathcal{E}\,\varphi,$$

where $\hat{\alpha} \equiv \{\alpha_1, \alpha_2, \alpha_3\}$ and β are 4×4 Dirac matrices

*We use occasionally Einstein's summation convention: Repeated indices as e.g $\gamma^\mu \hat{p}_\mu \Leftrightarrow \sum_{\mu=0}^{4} \gamma^\mu \hat{p}_\mu$

Mesons Mediating Nucleon-Nucleon Interactions

- **We learned that nucleons interact through exchange of q-q̄ pairs:**

π^+, π^0, π^-	—	**isovector, pseudoscalar;**
η	—	**isoscalar, pseudoscalar;**
ρ^+, ρ^0, ρ^-	—	**isovector, vector;**
ω	—	**isoscalar, vector;**
γ	—	**massless, vector;**

- **Using relativistic quantum field theory we may derive the Dirac equation for the nucleons in the presence of the exchange of mesons**

$$\{c\,\vec{\alpha}\cdot\hat{p} + \hat{V}(\vec{r})\;\mathbb{I}_4 + \beta\,[m_0c^2 + \hat{S}(\vec{r})]\}\psi_n = \mathcal{E}_n\psi_n,$$

Above: $\hat{V}$ and $\hat{S}$ are known functions originating from vector and scalar meson exchange, respectively (pseudo-scalars treated approx.)

A Mathematical Simplification: Pauli-Schrödinger Formalism

Standard Pauli-Schrödinger Reduction: Mathematics

- **Representing ψ in terms of 'big' and 'small' components:**

$$\psi \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix}; \quad \xi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}; \quad \eta \equiv \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}; \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

we may write two Schrödinger-like equations for spinors ξ and η

$$\hat{H}_\xi \, \xi_n = \mathcal{E}_n \, \xi_n \qquad \text{and} \qquad \hat{H}_\eta \, \eta_n = \mathcal{E}_n \, \eta_n$$

- **These Schrödinger-type Hamiltonians are non-linear in energy:**

$$\hat{H}_\xi \equiv (c\,\vec{\sigma}\cdot\hat{p}) \, \frac{1}{[\mathcal{E} + m_0c^2 - (\hat{V} - \hat{S})]} \, (c\,\vec{\sigma}\cdot\hat{p}) \; + \; [m_0c^2 + (\hat{V} + \hat{S})]$$

$$\hat{H}_\eta \equiv (c\,\vec{\sigma}\cdot\hat{p}) \, \frac{1}{[\mathcal{E} - m_0c^2 - (\hat{V} + \hat{S})]} \, (c\,\vec{\sigma}\cdot\hat{p}) \; - \; [m_0c^2 - (\hat{V} - \hat{S})]$$

Standard Pauli-Schrödinger Reduction - Properties

- **Eigen-energies $\mathcal{E}_n$ are common for both equations; they can be obtained by solving only one of them, usually for big component ξ_n**

$$\hat{H}_\xi \, \xi_n = \mathcal{E}_n \, \xi_n \quad \text{or} \quad \hat{H}_\eta \, \eta_n = \mathcal{E}_n \, \eta_n$$

- **The two Schrödinger-type equations are strictly equivalent to the original Dirac equation - there are no approximations here**

- The potentials depend only (!) on $\vec{r}$: $\hat{V} = \hat{V}(\vec{r})$ and $\hat{S} = \hat{S}(\vec{r})$

- The eigen-energies appear non-linearly → Bad News!

- Equations depend only on the sum and on the difference of the two original potentials - not on the individual ones → Interesting!
Very Interesting!

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$\langle \hat{S} \rangle \approx -400$ MeV and $\langle \hat{V} \rangle \approx +350$ MeV

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Position-Dependent Effective Mass: Definition

- **Let us recall the definition of the Pauli-Schrödinger Hamiltonian:**

$$\hat{H}_\xi \equiv \left(c\,\vec{\sigma}\cdot\hat{p}\right)\frac{1}{\{\mathcal{E}+m_0c^2-[\,\hat{V}-\hat{S}\,]\}}\left(c\,\vec{\sigma}\cdot\hat{p}\right)+\left[m_0c^2+[\,\hat{V}+\hat{S}\,]\right]$$

- By replacing $\mathcal{E}$ with $m_0c^2+\epsilon$, we may introduce the position-dependent effective mass $m^*(\vec{r})$

$$m^*(\vec{r}) \equiv \left\{m_0c^2-\tfrac{1}{2}[\hat{V}(\vec{r})-\hat{S}(\vec{r})]\right\}$$

and rewrite the denominator in the form:

$$\epsilon+2m_0c^2-[\hat{V}(\vec{r})-\hat{S}(\vec{r})]\equiv\epsilon+2m^*(\vec{r})$$

- Since $m_0c^2\approx 1000$ MeV and since inside the nucleus we have $\langle\frac{1}{2}[\hat{V}(\vec{r})-\hat{S}(\vec{r})]\}\rangle\approx 375$ MeV we find that $\langle 2m^*(\vec{r})\rangle\approx 750$ MeV

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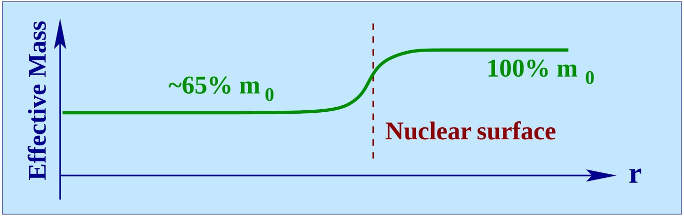
Position-Dependent Effective Mass: Estimates

- **Using the estimates $\langle\hat{S}\rangle \approx -400$ MeV and $\langle\hat{V}\rangle \approx +350$ we find**

$$\frac{1}{2m_0c^2 + \epsilon - (\hat{V} - \hat{S})} = \frac{1}{\epsilon + 2m^*} \simeq \frac{1}{2m^*}\left(1 - \frac{\epsilon}{2m^*}\right) \simeq \frac{1}{2m^*}$$

- In the above relations $2m^* \approx 1300$ MeV. For the levels close to the Fermi energy we have $|\epsilon| \sim (0 \text{ to } 10)$ MeV $\rightarrow \epsilon/2m^* \sim 0.01$ Thus Hamiltonians discussed are energy independent to 1% error



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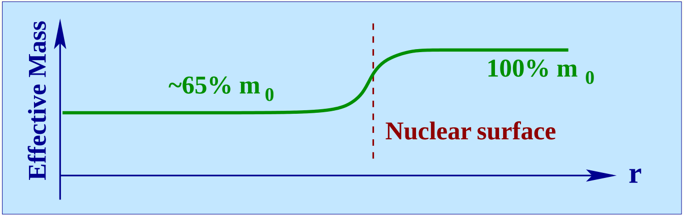


Linearized Pauli-Schrödinger Equation

- **The approximately linearised Pauli-Schrödinger equation then is:**

$$\Big\{(\vec{\sigma}\cdot\hat{p})\frac{1}{2m^*(\vec{r})}(\vec{\sigma}\cdot\hat{p}) + \underbrace{\big[\,\hat{S}(\vec{r})+\hat{V}(\vec{r})\,\big]}_{\sim -60\ \mathrm{MeV}}\Big\}\,\xi_n = \epsilon_n\,\xi_n$$

with the position-dependent effective mass:

$$m^*(\vec{r}) = \Big\{m_0c^2 - \tfrac{1}{2}\,\underbrace{\big[\,\hat{V}(\vec{r})-\hat{S}(\vec{r})\,\big]}_{\sim +750\ \mathrm{MeV}}\Big\}$$

- **The potential that binds the nucleons in the nucleus is the sum of the scalar- and vector-meson exchange contributions:**

$$\boxed{W(\vec{r}) \stackrel{df}{=} \hat{S}(\vec{r}) + \hat{V}(\vec{r}) \approx -60\ \mathrm{MeV}}$$

Form of the Generalized Kinetic Energy Operator

- **The operator quadratic in linear momenta can be transformed:**

$$(\vec{\sigma}\cdot\hat{p})\frac{1}{2m^*(\vec{r})}(\vec{\sigma}\cdot\hat{p}) = \frac{1}{2m^*(\vec{r})}\,\hat{p}^{\,2} + \hat{V}_{\vec{p}}(\vec{r},\hat{p}) + \hat{V}_{so}(\vec{r},\hat{p},\hat{s})$$

- **We recognise two new operators called 'potentials' despite the fact that they originate from the kinetic energy operator:**

$$\hat{V}_{so}(\vec{r},\hat{p},\hat{s}) \equiv \frac{2}{[2m^*(\vec{r})]^2}\{[\vec{\nabla}\underbrace{(\hat{V}(\vec{r})-\hat{S}(\vec{r}))}_{\sim 750\ \mathrm{MeV}}]\wedge\hat{p}\}\cdot\hat{s}$$

$$\hat{V}_{\hat{p}}(\vec{r},\hat{p}) \equiv \frac{-i\hbar}{[2m^*(\vec{r})]^2}[\vec{\nabla}\underbrace{(\hat{V}(\vec{r})-\hat{S}(\vec{r}))}_{\sim 750\ \mathrm{MeV}}]\cdot\hat{p}$$

- **In the following we find the interpretation of the above operators**

Physical Interpretation

Prediction of the Spin-Orbit Splitting Mechanism

- **The Simplest Case: Spherical Symmetry**

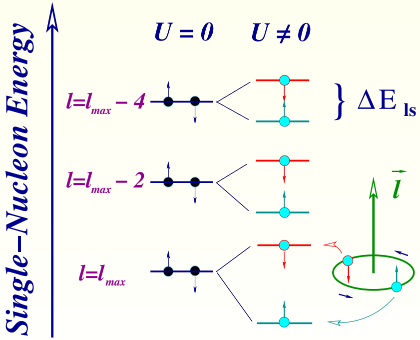
$$\mathbf{U}(\vec{\mathbf{r}}) \equiv \mathbf{U}(\mathbf{r}) \equiv \hat{\mathbf{V}} - \hat{\mathbf{S}} \;\rightarrow\; [\nabla \mathbf{U} \wedge \hat{\mathbf{p}}]\cdot\hat{\mathbf{s}} = \frac{1}{\mathbf{r}}\frac{\mathbf{dU}}{\mathbf{dr}} \overbrace{(\vec{\mathbf{r}} \wedge \hat{\mathbf{p}})}^{\hat{\ell}} \cdot \hat{\mathbf{s}} = \frac{1}{\mathbf{r}}\frac{\mathbf{dU}}{\mathbf{dr}} \hat{\ell}\cdot\hat{\mathbf{s}}$$

$$\uparrow (\ell, s) \uparrow \;: \quad \langle \hat{\ell} \cdot \hat{s} \rangle = +\tfrac{1}{2}\ell$$

$$\uparrow (\ell, s) \downarrow \;: \quad \langle \hat{\ell} \cdot \hat{s} \rangle = -\tfrac{1}{2}(\ell + 1)$$

Observe the correct sign of $\Delta E_{\ell s}$:

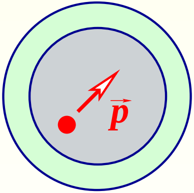
$$U = V - S > 0 \;\rightarrow\; \frac{dU}{dr} < 0$$

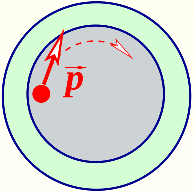


Potentials V_{so} and V_p - An Illustration

- Potential $\hat{V}_p$ is responsible for 'de-acceleration' proportional to $\hat{p}$
- Both potentials stop acting at the limit $\vec{v} \sim \vec{p}/m_0 \to 0$ ('kinetic')

$$V_p \sim \frac{dU}{dr}\left(\frac{\vec{r}}{r}\right)\cdot\hat{p} \qquad\qquad V_{so} \sim \frac{1}{r}\frac{dU}{dr}\,\hat{\ell}\cdot\hat{s}$$





Potential V_p: It is transparent to the circular motion, and it is independent of spin

Potential V_{so}: It is indifferent to the radial motion while its action depends on spin

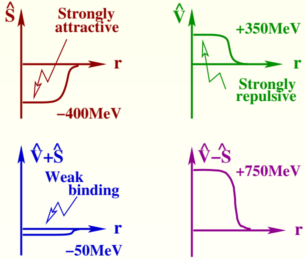
Realistic $\hat{V}$ and $\hat{S}$ Potentials

- **Observe a paradox: a very strong attractive potential $\hat{S}$ and a very strong repulsive potential $\hat{V}$, sum up to only very weak total nucleonic binding: $\hat{V}+\hat{S}$**

Observe that the very strong, positive [$\hat{V}$-$\hat{S}$] term contributes:

a. Only through the gradient in the spin-orbit and in the linear momentum potentials;

b. Preceded by the 'minus' sign in the definition of the effective mass .



What Did We Learn About Nuclear Secrets So Far?

- **The nuclear interactions originate from the exchange of mesons**
- **The scalar mesons contribute to a strong attraction (∼400 MeV)**
- **The vector mesons contribute to a strong repulsion (∼350 MeV)**
- **The nucleons in nuclei are very weakly bound (∼ -10 to 0 MeV)**
- **From experiment: p-p and n-n are not bound, p-n: just one state**
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- **There exist Momentum and Spin-Orbit 'potentials'. Their origin:**

$$\textbf{Kinetic Energy Operator:}\quad \hat{t} \equiv (\vec{\sigma}\cdot\hat{p}\,)\,\frac{1}{2m^*(\vec{r})}\,(\vec{\sigma}\cdot\hat{p}\,)$$

- **Tangential orbits couple with spin: bouncing-ball orbits do not!**
- **Parallel $\vec{\ell}$ and $\vec{s}$ coupling is privileged; anti-parallel 'discouraged'**

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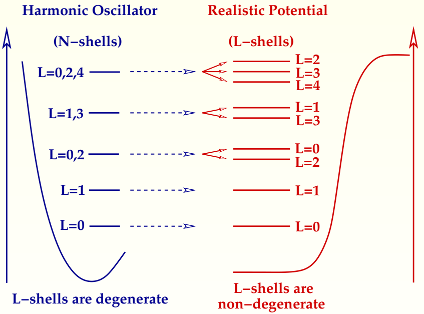
Part IV

Mean-Field Theory: Link with Experiment

Quantum Mechanics: Memory Refreshing Facts

• In the harmonic-oscillator case there exists a special symmetry that makes L-shells degenerate; for realistic nuclear potentials this symmetry does not hold anymore. Observe: N-shells and L-shells:



• Levels E_{LM} are M-degenerate, $E_{LM} = E_{LM'}$ $(-L \leq M, M' \leq +L)$. This 'magnetic' degeneracy results from the spherical symmetry

Quantum Mechanics: Memory Refreshing Facts (II)

- **It is well known from elementary quantum mechanics that for hamiltonians with spherical symmetry:**

$$[\hat{H},\hat{j}^2]=0,\quad [\hat{H},\hat{j}_z]=0,\quad [\hat{H},\hat{\ell}^2]=0,\quad [\hat{H},\hat{s}]=0,\quad \hat{j}\equiv\hat{l}+\hat{s}$$

- **The solutions are simultaneous eigenstates of $\hat{H}$, $\hat{j}^2$, $\hat{j}_z$ and $\hat{\ell}^2$**

$$\hat{H}\psi_{n;j\ell m}=E_{n;j\ell m}\psi_{n;j\ell m}$$

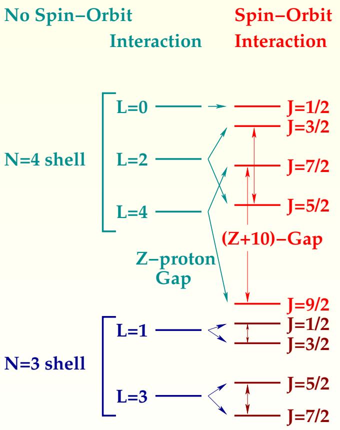
- **This allows to introduce the spectroscopic notation based on:**

$$\left.\begin{array}{ccccccccc} \ell & = & 0 & 1 & 2 & 3 & 4 & 5 & 6\ldots \\ & & s & p & d & f & g & h & i\ldots \end{array}:\right\}\quad n_r\ell_j$$

for instance $1s_{1/2}$, $2d_{5/2}$, $3p_{1/2}$, $1i_{13/2}$ etc.

Spin-Orbit Splitting and Nobel Prize

- **Left: results with no-spin-orbit potential; Right: with the spin-orbit potential**
- **Vertical arrows denote the so-called spin-orbit splitting**
- **In atomic nuclei this splitting is very large, ejecting the lowest energy, the highest-J orbital, to the (N-1st)-shell below**
- **The ejected orbitals are called 'intruders'; for their discovery M. Göppert-Mayer and J. Jensen received the Nobel Prize in 1963**



Spin−Orbit Splitting Mechanism

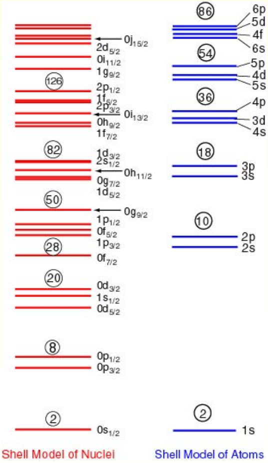
Spin-Orbit Splitting and Nobel Prize

• **At the discovery time, the mechanism of spin-orbit splitting was not trivial at all: observe the differences between nuclear and atomic cases**

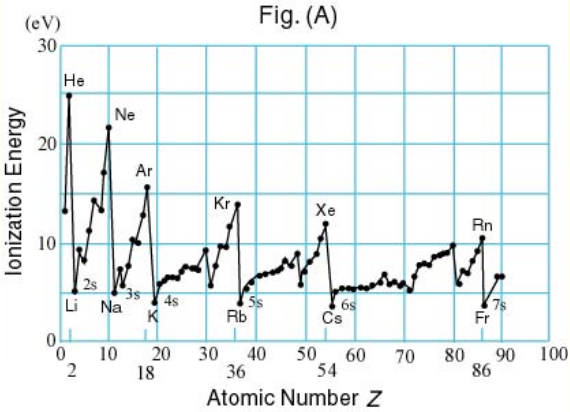
• **The 1963 Nobel Prize for explanation of the nuclear Göppert-Mayer and Johannes Jensen [together with Eugene Wigner]**

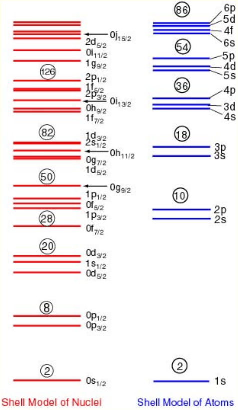
• **Today we know that the spin-orbit potential describing the magic numbers is in fact spin-orbit kinetic energy**

• **Gaps in the spectra are measurable quantities; measurements fully confirm the discussed mechanism**



Energy Gaps and Experimetal Confirmation





- **Correlation: Maxima in ionization energy and the big gaps**

Part V

The New Theory of Nuclear Stability

The New Theory of Nuclear Stability

.. will be presented some other time ...

Please come to Strasbourg again

Very Schematically: What the New Theory Is About?

"New Theory of Nuclear Stability"

Very Schematically: What the New Theory Is About?

"New Theory of Nuclear Stability"

(Its Principles and Main Lines)

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Nuclear Stability within the Mean–Field Theory

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Nuclear Stability within the Mean–Field Theory → **Large gaps in the single particle spectra**

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→ **Degeneracies of s.p. levels (exact or approximate)**

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Generalized Formulation of the rules of Nuclear Stability

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Generalized Formulation of the rules of Nuclear Stability → 1. Each symmetry generates its proper Magic Numbers and the induced Series of Magic Gaps

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Generalized Formulation of the rules of Nuclear Stability →

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2. The symmetry effect is stronger if the group has more irreducible representations

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Generalized Formulation of the rules of Nuclear Stability →

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3. The symmetry effect is stronger, if irreducible repres. have larger dimensions

Symmetries, Representations and Degeneracies

The group-analysis needed is a standard textbook information (almost)

No.	Group	No. Irr.	Dimensions
01.	O_h^D	6	4 x 2D and 2 x 4D
02.	O^D	3	2 x 2D and 1 x 4D
03.	T_d^D	3	2 x 2D and 1 x 4D
04.	C_{6h}^D	12$\longrightarrow$ 6	12 x 1D
05.	D_{6h}^D	6	6 x 2D
06.	T_h^D	6	6 x 2D
07.	D_{4h}^D	4	4 x 2D
–	D_{2h}^D	2	2 x 2D (reference)

Table: *According to the above criteria our first choice are* O_h^D *and* T_d^D

Part V

Unitary Group and Many-Body Hamiltonians

Groups Composed of Unitary Matrices U_n and SU_n

- Groups $U_n \leftrightarrow n \times n$ unitary matrices; $SU_n \leftrightarrow [\det(U) = 1]$
- It turns out that these matrices can be expressed as

$$U = \exp\Big[-i \sum_{\alpha\beta} p_{\alpha\beta}\, \hat{g}_{\alpha\beta} \Big], \quad p_{\alpha\beta} \leftrightarrow \text{real parameters} \qquad \text{(A)}$$

- One can show that operators (generators) $\hat{g}_{\alpha\beta}$ satisfy

$$[\hat{g}_{\alpha\beta}, \hat{g}_{\gamma\delta}] = \delta_{\beta\gamma}\, \hat{g}_{\alpha\delta} - \delta_{\alpha\delta}\, \hat{g}_{\gamma\beta}$$

- We say that the fixed operators $\{\hat{g}_{\alpha\beta}\}$ generate all the unitary matrices for the infinity of varying real parameters $\{p_{\alpha\beta}\}$

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N-Body Systems with Two-Body Interactions

- **Consider an N-particle system with a two-body Hamiltonian**

$$\hat{H} = \sum_{\alpha\beta} h_{\alpha\beta}\, c^+_\alpha c_\beta + \frac{1}{2} \sum_{\alpha\beta} \sum_{\gamma\delta} v_{\alpha\beta;\gamma\delta}\, c^+_\alpha c^+_\beta\, c_\delta c_\gamma$$

- **Introduce operators**

$$\hat{N}_{\alpha\beta} \overset{\text{df.}}{=} c^+_\alpha c_\beta$$

- **It is easy to verify that**

$$[\,\hat{N}_{\alpha\beta}, \hat{N}_{\gamma\delta}] = \delta_{\beta\gamma}\hat{N}_{\alpha\delta} - \delta_{\alpha\delta}\hat{N}_{\gamma\beta}$$

compare

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N-Body Hamiltonians and U_n-Group Generators

- **N-Body Hamiltonians are functions of U_n-group generators**

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- **Two-body interactions lead to quadratic forms of $\hat{N}_{\alpha\beta} = c^+_\alpha c_\beta$, three-body interactions to the cubic forms of $\hat{N}_{\alpha\beta}$, etc.**
- **Hamiltonians of the N-body systems can be diagonalised within bases of the irreducible representations of the U_n groups**
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Consequences in Terms of Sub-Groups

Group U_n has numerous sub-groups: U_m and SU_m with $m < n$, similarly O_m, SO_m and in particular R_3 and all the point groups

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Subgroup Structure Can Be Very, Very Rich ...

32 Point Groups: Subgroups

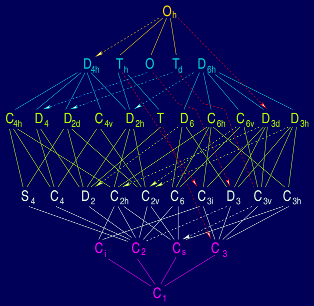

Figure: *Richness of the sub-group structures at the end of chain...*

Dashed lines indicate that the subgroup marked is not invariant

Trivial groups are denoted here:

$C_1 = \{\mathbb{I}\}$, $C_s = \{\mathbb{I}, \hat{\sigma}\}$,
$C_i = \{\mathbb{I}, \hat{\pi}\}$, $C_2 = \{\mathbb{I}, \hat{R}_2\}$,
$C_3 = \{\mathbb{I}, \hat{R}_3, \hat{R}_3^2\}$

Here we show the structure only at the very end of the U_n chain - it helps imagining how rich the full group structure is ...

Part VI

Annexe 1: Identical Particles and Pauli Principle

About Identical Particles

• Consider a many-body system composed of n identical particles.

We use position, linear momentum and spin, $\hat{r}$, $\hat{p}$, $\hat{s}$, to describe a particle $\hat{x} \equiv \{\hat{r}, \hat{p}, \hat{s}\}$. The Hamiltonian $\hat{H} = \hat{H}(\hat{x}_1, \hat{x}_2, \ldots \hat{x}_n)$ must be symmetric under any permutation

$$\hat{\mathcal{P}}_{ij}\, \hat{H}(\hat{x}_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots \hat{x}_n)\, \hat{\mathcal{P}}_{ij}^{-1} \stackrel{df}{=} \hat{H}(\hat{x}_1 \ldots \hat{x}_j \ldots \hat{x}_i \ldots \hat{x}_n) = \hat{H}(\hat{x}_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots \hat{x}_n)$$

and it follows: $\boxed{\hat{\mathcal{P}}_{ij}\, \hat{H}\, \hat{\mathcal{P}}_{ij}^{-1} = \hat{H} \quad \rightarrow \quad [\hat{\mathcal{P}}_{ij}, \hat{H}] = 0, \quad \forall\, i \neq j \leq n\,.}$

• Conclusions: 1. Both observables $\hat{\mathcal{P}}_{ij}$ and $\hat{H}$ can be diagonalized simultaneously; 2. Eigenvalues of $\hat{\mathcal{P}}_{ij}$ are constants of motion.

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About Identical Particles (2)

- Since $\hat{\mathcal{P}}_{ij}^2 = 1$ it follows that in $\boxed{\hat{\mathcal{P}}_{\mathbf{ij}}\boldsymbol{\Psi} = \mathbf{p_{ij}}\boldsymbol{\Psi},}$ we must have

$$\mathbf{p_{ij}^2 = 1 \quad \rightarrow \quad p_{ij} = \pm 1}$$

This implies that identical particles are either

$$\boxed{\textbf{Fermions :} \quad \hat{\mathcal{P}}_{\mathbf{ij}}\, \boldsymbol{\Psi}_{\mathbf{n_1,\, \dots\, n_i,\, \dots n_j,\, \dots\, n_n}} = -\boldsymbol{\Psi}_{\mathbf{n_1,\, \dots\, n_i,\, \dots n_j,\, \dots\, n_n}}, \quad \forall\ \mathbf{i, j}}$$

or

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<u>and this all life-long for all identical particles of a given type</u>

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About Identical Particles: Pauli Principle

- We say that the wave-functions for identical Fermions are totally anti-symmetric and those for Bosons are totally symmetric
- By setting for Fermions $i = j$ (two identical states) we have

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- Thus the wave-function satisfies generally $\Psi = -\Psi$ or $\Psi \equiv 0$.

Pauli: We must not have two identical Fermions in the identical quantum states

Physical many-body states are either totally symmetric or totally anti-symmetric

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Fermion Wave-Functions: Anti-Symmetrisation

• Let us begin by posing a certain elementary problem that some of you know already how to tackle:

What is the structure of $s = \frac{1}{2}$ two-particle wave functions at $\vec{r}_1$ and $\vec{r}_2$?

• The wave functions depend on the spatial parts $\varphi_\alpha(\vec{r}\,)$ and $\varphi_\beta(\vec{r}\,)$ and the spin part χ_{s,s_z}: the total wave functions must be antisymmetric:

$$\boldsymbol{\Psi}_{\alpha\beta} \sim \textbf{Anti-symm.}[\varphi_\alpha(\vec{\mathbf{r}}_1\,), \varphi_\beta(\vec{\mathbf{r}}_2\,)] \times \textbf{Symm.}[\chi_{\mathbf{s},\mathbf{s}_{z,1}}, \chi_{\mathbf{s},\mathbf{s}_{z,2}}]$$

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What is the structure of $s = \frac{1}{2}$ two-particle wave functions at $\vec{r}_1$ and $\vec{r}_2$?

• The wave functions depend on the spatial parts $\varphi_\alpha(\vec{r}\,)$ and $\varphi_\beta(\vec{r}\,)$ and the spin part χ_{s,s_z}: the total wave functions must be antisymmetric:

$$\boldsymbol{\Psi_{\alpha\beta} \sim \textbf{Anti-symm.}[\varphi_\alpha(\vec{r}_1\,), \varphi_\beta(\vec{r}_2\,)] \times \textbf{Symm.}[\chi_{s,s_{z,1}}, \chi_{s,s_{z,2}}]}$$

$$\boldsymbol{\Psi_{\alpha\beta} \sim \textbf{Symm.}[\varphi_\alpha(\vec{r}_1\,), \varphi_\beta(\vec{r}_2\,)] \times \textbf{Anti-symm.}[\chi_{s,s_{z,1}}, \chi_{s,s_{z,2}}]}$$